Appendix A. Adaptive Meshes

Appendix A.1. The Galerkin finite element method

Let us define the test functions space

$$V = \left\{ v \, : \, \int_{\Omega} (|v'|^2 + v^2)\, dx < \infty, \quad v(0) = v(1) = 0 \right\}$$

and we take the trial functions space $V^1 = V$. We apply the standard Galerkin method to problem (2.4) and integrate the differential equation by parts, this gives the variational problem formulation: find $u \in V$ such that

$$\int_0^1 \left(\varepsilon^2 u'v' + quv\right)\, dx = \int_0^1 fv\, dx \quad \forall\ v \in V. \tag{A.34}$$

We consider piecewise linear trial and test functions. Let $W_h \, : \, 0 = x_0 < x_1 < \ldots < x_{N-1} < x_N = 1$ be a partition of $[0, 1]$ and let $V_h \subset V$ be the corresponding finite element space of continuous piecewise linear functions. The finite element method is obtained by applying the Galerkin method to (A.34) with the finite dimensional space $V_h \subset V$ replacing V and reads: find $U \in V_h$ such that

$$\int_0^1 \left(\varepsilon^2 U'v' + qUv\right)\, dx = \int_0^1 fv\, dx \quad \forall\ v \in V_h. \tag{A.35}$$

Let φ_i be nodal basis functions

$$\varphi_i = \begin{cases} (x - x_{i-1})/h_{i-0.5}, & x \in [x_{i-1}, x_i) \\ (x_{i+1} - x)/h_{i+0.5}, & x \in [x_i, x_{i+1}] \\ 0, & \text{otherwise} \end{cases},$$

where $h_{j-0.5}{=}x_j{-}x_{j-1}$, then any $v \in V_h$ can be written as $v(x){=}\sum\limits_{j=1}^{N-1} \mu_j \varphi_j(x)$, where μ_j are coefficients. We write U in terms of nodal basis functions φ_j, $j = 1, \ldots N-1$,

$$U(x) = \sum_{j=1}^{N-1} U_j \varphi_j(x), \tag{A.36}$$

and substitute this expression into (A.35). Choosing v in (A.35) to be $\varphi_i, i = 1, \dots, N-1$, we obtain the system of equations for U_j which is tridiagonal. Solving this system gives $U(x)$ from (A.36).

Appendix A.2. Mesh generation using duality-based technique

In this section, we apply the well-known duality-based error estimates for the problem (2.4). A standard technique described e.g. in (Eriksson, Estep, Hansbo, Johnson 1995) for general second order boundary value problems is used. We make no attempt to construct special uniform in small parameter ε error estimates, since boundary layers should be resolved by special layer-resolving transformations based on adaptive meshes. The error is estimated in the L_2 and energy norms.

Error estimation in the L_2 norm

Let us define $Au := -\varepsilon^2 u'' + qu$, then problem (2.4) can be written as

$$\begin{cases} Au = f, & \text{in } x \in (0,1), \\ u(0) = 0, & u(1) = 0. \end{cases}$$

A scalar product and the L_2 norm are defined as

$$(u, v) = \int_0^1 uv \, dx, \quad \|u\| = \sqrt{(u,u)}.$$

Let us denote the error $e = u - U$. In order to estimate this error, we find z as a solution of the dual problem

$$\begin{cases} A^* z = e, & \text{in } x \in [0,1], \\ z(0) = 0, & z(1) = 0. \end{cases} \tag{A.37}$$

In our case $A^* = A$. Multiplying (A.37) by e, integrating by parts and considering $v = \pi_h z$ in (A.35), where $\pi_h z \in V_h$ is the piecewise linear interpolant of z, we get the estimate

$$(e, e) \leq \left\| h^2 (qU - f) \right\| \left\| h^{-2} (z - \pi_h z) \right\|. \tag{A.38}$$

Here $h = h(x) = x_i - x_{i-1}$ in $x_{i-1} < x \leq x_i$. The error of the the piecewise linear interpolant $\pi_h z$ can be estimated as (Eriksson, Estep, Hansbo, Johnson

1995)

$$\left\|h^{-2}(z-\pi_h z)\right\| \leq C_i \left\|z''\right\|. \tag{A.39}$$

To estimate the term $\|z''\|$ in (A.39) we consider the following problem

$$\begin{cases} Aw = \psi, & \text{in } x \in [0,1] \\ w(0) = 0, & w(1) = 0, \end{cases} \tag{A.40}$$

where $\psi = \psi(x) \in V$ are functions that represent all possible error functions $e = e(x)$. Then the strong stability factor S is defined by

$$S = \max_{\psi \in V} \frac{\|w''\|}{\|\psi\|}.$$

Consider there $\psi = e$, then from (A.37) we get that $w = z$ and obtain the estimate

$$\left\|z''\right\| \leq S \left\|e\right\|. \tag{A.41}$$

Finally, from (A.38), (A.39) and (A.41) we obtain the aposteriori error estimate in the L_2 norm

$$\|e\| \leq \bar{C}_i \left\|h^2 q(U-f)\right\|. \tag{A.42}$$

Here $\bar{C}_i = SC_i$ is a constant and it has no effect on construction of adaptive meshes.

Error estimation in the energy norm

Let us define the energy norm $\|e'\| = \sqrt{(e', e')}$. Replacing $v \in V$ with an error function $e \in V$ in variational problem (A.34), considering $v = \pi_h e$ in (A.35), where $\pi_h e \in V_h$ is a piecewise linear interpolant of e, integrating by parts and using the Cauchy inequality, gives

$$\varepsilon^2 \left\|e'\right\|^2 \leq \|h(qU-f)\| \left\|h^{-1}(e-\pi_h e)\right\|. \tag{A.43}$$

The interpolation error can be estimated as (Eriksson, Estep, Hansbo, Johnson 1995)

$$\left\|h^{-1}(e-\pi_h e)\right\| \leq \widetilde{C}_i \left\|e'\right\|,$$

where $\widetilde{C}_i$ is a constant. Using this inequality in (A.43) we get an aposteriori error estimate in the energy norm

$$\left\|e'\right\| \leq \frac{\widetilde{C}_i}{\varepsilon^2} \|h(qU-f)\|. \tag{A.44}$$

We note that in both aposteriori error estimates (A.42) and (A.44) the residual $R(U) = qU - f$ of the FE scheme multiplied by h^p defines the contribution to the total error from the element $[x_{j-1}, x_j]$.

Adaptive mesh generation

The adaptive mesh is obtained by solving the minimization problem

$$\min_{x_1,\dots,x_{N-1}} \sum_{j=1}^{N} r_{j-0.5}, \quad \sum_{j=1}^{N} h_{j-0.5} = 1, \tag{A.45}$$

where $r_{j-0.5}$ is a local error estimate on the interval $[x_{j-1}, x_j]$. To minimize the total error the standard local error equidistribution technique is applied for a given monitoring function (Eriksson, Estep, Hansbo, Johnson 1995). It reads: find $h_{j-0.5}$, $j = 1, \dots, N$, such that $r_{j-0.5} = const$, $j = 1, \dots, N$.

Using aposteriori error estimates (A.42), (A.44) we get the following error equidistribution problem

$$r_{j-0.5} = h_{j-0,5}^{2p} \int_{x_{j-1}}^{x_j} (qU - f)^2 \, dx = const, \quad j = 1, 2, \dots, N, \tag{A.46}$$

here $p = 1$ for the energy norm and $p = 2$ for the L_2 norm, respectively. Problem (A.46) can be solved by the inverse interpolation method.

Appendix A.3. Comparison with apriori adaptive meshes

In this section we compare the adaptive mesh based on aposteriori error estimates with the well known Shishkin and Bakhvalov meshes.

Comparison with the Bakhvalov mesh

The Bakhvalov mesh (Bakhvalov 1969) is generated by equidistributing

$$\int_{x_{j-1}}^{x_j} M_B(x) \, dx = const, \quad j = 1, 2, \dots, N \tag{A.47}$$

the monitoring function

$$M_B(x) = \max\left\{\alpha, e^{\left(-\frac{\rho x}{\sigma\varepsilon}\right)}, e^{\left(-\frac{\rho(1-x)}{\sigma\varepsilon}\right)}\right\}, \tag{A.48}$$

where $\varrho = \min_{x\in[0,1]} \sqrt{q(x)}$, σ is a freely defined parameter, and α is a regularization parameter.

The information on function f is not used in the Bakhvalov mesh. Also, we note, that in the Bakhvalov mesh the function $q(x)$ is approximated by the constant $\varrho = \min_{x\in[0,1]} \sqrt{q(x)}$.

In order to find a relation between duality-based a posteriori mesh and the Bakhvalov mesh we consider the problem

$$\begin{cases} -\varepsilon^2 u'' + \rho^2 u = \bar{f}, & \text{in } x \in (0,1), \\ u(0) = 0, \quad u(1) = 0, \end{cases} \tag{A.49}$$

where $\bar{f} = const$. To obtain an analytical formula of the adaptive meshes in the L_2 and energy norms we use the exact solution of (A.49) instead of the Galerkin approximation (A.46):

$$u(x) = \frac{\bar{f}}{\rho^2} \frac{-e^{\left(\frac{\rho x}{\varepsilon}\right)} + e^{\left(-\frac{\rho x}{\varepsilon}\right)} - e^{\left(\frac{\rho(1-x)}{\varepsilon}\right)} + e^{\left(-\frac{\rho(1-x)}{\varepsilon}\right)} - e^{\left(-\frac{\rho}{\varepsilon}\right)} + e^{\left(\frac{\rho}{\varepsilon}\right)}}{e^{\left(\frac{\rho}{\varepsilon}\right)} - e^{\left(-\frac{\rho}{\varepsilon}\right)}}.$$

Then the residual of the Galerkin scheme can be computed as

$$R(u) = \frac{\bar{f}}{e^{\left(\frac{\rho}{\varepsilon}\right)} - e^{\left(-\frac{\rho}{\varepsilon}\right)}} \left(-e^{\left(\frac{\rho x}{\varepsilon}\right)} + e^{\left(-\frac{\rho x}{\varepsilon}\right)} - e^{\left(\frac{\rho(1-x)}{\varepsilon}\right)} + e^{\left(-\frac{\rho(1-x)}{\varepsilon}\right)} \right).$$

We are interested to analyze the adaptive mesh only in the neighbourhood of boundary layers $0 \le x \le c\varepsilon/\rho$ and $0 \le 1 - x \le c\varepsilon/\rho$, thus it is sufficient to take the following asymptotical approximation of the residual

$$\widetilde{R}(u) = -\bar{f} \left(e^{-\frac{\rho x}{\varepsilon}} + e^{-\frac{\rho(1-x)}{\varepsilon}} \right). \tag{A.50}$$

Then from (A.46) we get the following error equidistribution problem

$$h_{j-0,5}^{2p} \int_{x_{j-1}}^{x_j} \left(e^{-\frac{\rho x}{\varepsilon}} + e^{-\frac{\rho(1-x)}{\varepsilon}} \right)^2 dx = const, \quad j = 1, 2, \ldots, N. \tag{A.51}$$

First, let us consider a left boundary layer at $x = 0$. Since $\varepsilon \ll 1$, equations (A.51) can be simplified to

$$h_{j-0,5}^{2p} \int_{x_{j-1}}^{x_j} e^{-2\rho x/\varepsilon}\, dx = const, \quad j = 1, 2, \ldots, N/2.$$

Applying the standard midpoint rule for numerical integration after simple computations we derive the mesh equidistribution equations in the following form

$$\int_{x_{j-1}}^{x_j} e^{-\frac{2\rho x}{\varepsilon(2p+1)}}\, dx = const, \quad j = 1, 2, \ldots, N/2. \tag{A.52}$$

A boundary layer at $x = 1$ is considered similarly. Thus the duality based adaptive mesh can be obtained by using the monitoring function

$$M(x) = \begin{cases} e^{-\frac{2\rho x}{\varepsilon(2p+1)}}, & \text{in } x \in [0, 0.5), \\ e^{\frac{2\rho(x-1)}{\varepsilon(2p+1)}}, & \text{in } x \in [0.5, 1]. \end{cases} \tag{A.53}$$

By taking $\sigma = \frac{2p+1}{2}$ in (A.53) we get the monitoring function (A.48), which was used to construct the Bakhvalov mesh. In addition we have derived a rule to define parameter σ for the Bakhvalov mesh: $\sigma = \frac{5}{2}$, if the global error is estimated in the L_2 norm, and $\sigma = \frac{3}{2}$, if the energy norm is used.

At the end of this subsection we present explicit formulas for the adaptive mesh, generated by the monitoring function (A.53). From (A.52) we get a system of nonlinear equations

$$\int_0^{x_j} e^{\left(-\frac{\rho x}{\sigma\varepsilon}\right)}\, dx = \frac{2j}{N} \int_0^{1/2} e^{-\left(\frac{\rho x}{\sigma\varepsilon}\right)}\, dx, \quad j = 1, \ldots, N/2,$$

from which x_j is defined by:

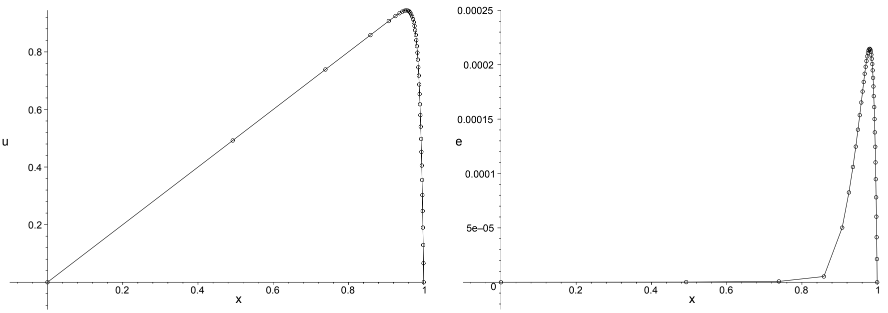
$$x_j = -\frac{\sigma\varepsilon}{\rho} \ln\left(1 - \frac{2j}{N}\left(1 - e^{-\frac{\rho}{2\sigma\varepsilon}}\right)\right), \quad j = 1, \ldots, N/2. \tag{A.54}$$

Since the mesh is symmetric with respect to point $x = 0.5$, then $x_j = x_{N-j}$ for $j = N/2 + 1, \ldots, N$.

The Bakhvalov mesh is exponentially fitted at boundary layers. In Fig. 2.14*a* an example of the Bakhvalov mesh is shown for $f = q = k = p = 1$, $\varepsilon = 0.05$, discretization nodes are marked by vertical lines, $N = 40$.

Numerical experiments

In this section we investigate how assumptions used to construct the Shishkin mesh affect the accuracy of the discrete solution. For error computations we have computed numerical solutions by taking a very big number of mesh points $N = 16000$ of the Bakhvalov mesh, the mesh regularization parameter α in formula (A.48) is defined by the technique offered by Beckett, Mackenzie



(a) **(b)**

Fig. A.6. Results for Example 1 with $\varepsilon = 0.01$: a) an adaptive mesh based on a posteriori error estimates and the Galerkin solution, b) the global error $|e(x)|$ for problem (A.55)

(2001). In all computations we use $\sigma = \frac{5}{2}$. Computational experiments are done for the Shishkin mesh and adaptive mesh based on a posteriori error estimate in the L_2 norm. Nodal errors $e(x) = u(x) - U(x)$ in the maximum norm are also presented.

Example 1. Consider the following singular problem

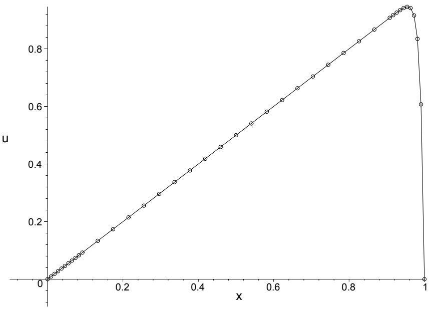
$$\begin{cases} -\varepsilon^2 u'' + u = x, & \text{in } x \in (0,1), \\ u(0) = 0, \quad u(1) = 0. \end{cases} \tag{A.55}$$

The solution of this problem do not have boundary layer singularity on the left side of interval. The adaptive mesh based on a posteriori error estimate, the Galerkin solution and global errors of this solution are presented in Fig. Fig. A.6. The most of mesh points are distributed on the right boundary layer at $x = 1$ and they are adapted to local values of f.

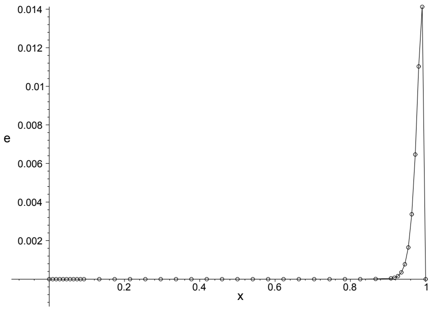
At it was stated above, the Shishkin mesh does not depend on local values of the function $f(x)$ and the mesh points are adapted to both boundary layers, though the left one is not presented in this example. We see in Fig. Fig. A.7, that for the Shishkin mesh the largest errors of the Galerkin solution are distributed near the right boundary layer.

Next we have solved problem (A.55) and computed the error by using both meshes with different numbers of nodes N. The errors in the L_2 norm and the experimental convergence rates are given in Table A.3.

It follows from the presented results, that the adaptive mesh based on



(a)



(b)

Fig. A.7. Results for Example 1 with $\varepsilon = 0.01$: a) the Shishkin mesh and the Galerkin solution, b) the global error $|e(x)|$ for problem (A.55)

Table A.3. Errors of the Galerkin solution in the L_2 norm and the experimental convergence rates for problem (A.55)

n	the Shishkin mesh	rate	adaptive mesh	rate
20	1.08E-2	--	1.81E-4	--
40	4.36E-3	1.31	3.78E-5	2.26
80	1.58E-3	1.46	8.70E-6	2.12
160	5.36E-4	1.56	2.09E-6	2.06
320	1.74E-4	1.63	5.11E-7	2.03
640	5.45E-5	1.67	1.26E-7	2.02

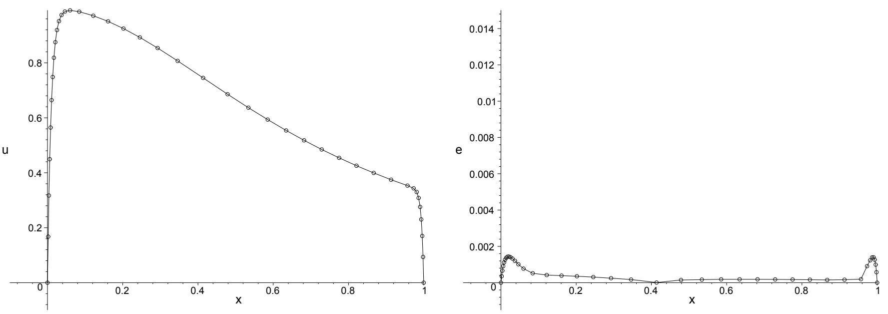
a posteriori error estimates gives more accurate Galerkin solution than the Shishkin mesh. However the difference between both results is not big.

Example 2. In this example we consider a singular problem with different thicknesses of boundary layers:

$$\begin{cases} -\varepsilon^2 u'' + (1+2x^2)u = 1, & \text{in } x \in (0,1), \\ u(0) = 0, \quad u(1) = 0. \end{cases} \tag{A.56}$$

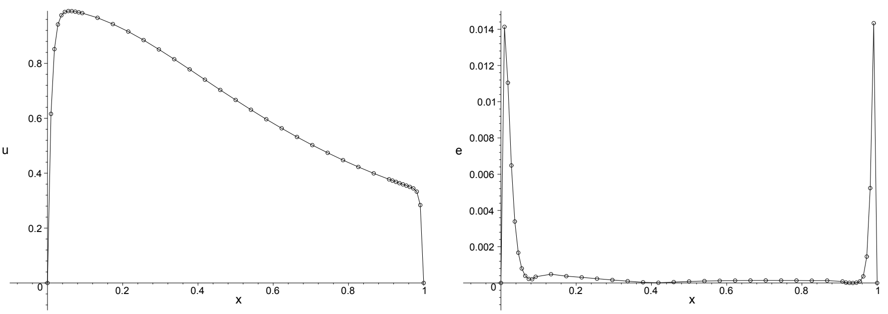
The thickness of the right boundary layer is smaller than the thickness of the left boundary layer. The adaptive mesh based on a posteriori error estimate, the Galerkin solution and global errors of this solution are presented in Fig. Fig. A.8. We see that global error of the Galerkin solution is well balanced at both boundary layers.

The Shishkin mesh does not depend on local values of the function $q(x)$, thus the thickness of both boundary layers is taken the same in this case. We see in Fig. Fig. A.9, that for the Shishkin mesh the largest errors of the Galerkin



(a) **(b)**

Fig. A.8. Results for Example 2: a) an adaptive mesh based on a posteriori error estimates and the Galerkin solution, b) the global error $|e(x)|$ for problem (A.56)



(a) **(b)**

Fig. A.9. Results for Example 2: a) the Shishkin mesh and the Galerkin solution, b) the global error $|e(x)|$ for problem (A.56)

solution are distributed near the right boundary layer, where the real thickness is smaller than defined by the mesh and some mesh points are redundant there.

Next we have solved problem (A.56) and computed the error by using adaptive and the Shishkin meshes with different numbers of nodes N. The errors in the L_2 norm and the experimental convergence rates are given in Table A.4.

Table A.4. Errors of the Galerkin solution in the L_2 norm and the experimental convergence rates for problem (A.56)

n	the Shishkin mesh	rate	adaptive mesh	rate
20	1.28E-2	–	1.82E-3	–
40	5.33E-3	1.27E	3.31E-4	2.46E
80	1.96E-3	1.44E	7.02E-5	2.24E
160	6.70E-4	1.55E	1.66E-5	2.08E
320	2.18E-4	1.62E	4.14E-6	2.01E
640	6.84E-5	1.67E	1.04E-6	2.00E