OPTIMIZATION-BASED ELASTIC-PLASTIC ANALYSIS OF STEEL FRAMES BY VOLUMETRIC PLASTICITY CONCEPT

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Abstract. Structures composed of physical nonlinear finite elements under bending and compression or tension are considered in this paper. Material nonlinearity is considered as linearly hardened. In case of material hardening, plastic strains do not concentrate in one point but distribute in the certain volumes of finite element. Volumes of plastic strains zones in frame structure elements impact on elasticity modules of these elements sections by decreasing them. Technique of such elasticity modules decrease in finite elements sections is suggested. To realize such structure analysis, a treatment of strains in mathematical model is changed. Now strains are treated not as rotation angles and elongations of finite elements, but as longitudinal strains. Mathematical model including above mentioned modifications is presented. Solving algorithm based on a modified Newton-Raphson method is particularly explained and employed for numerical example.

Keywords: frames analysis, finite element method, material linear hardening, plastic zones development, modified Newton-Raphson method, incremental loading.

Introduction

An elastic-perfectly plastic material is analyzed in most papers (Jones et al. 2001; Jaras and Kačianauskas 2003; Čyras and Atkočiūnas 1984) related to elastic plastic structures analysis or optimization. Meanwhile, the real stress-strain dependency more or less differs from idealized one. Therefore, structure behavior will be described more exactly when material is treated as linearly hardened (Sawko 1964). In this case, plastic strains do not concentrate in one point (i.e. in dangerous section), but distribute in the certain volumes of finite element. Thus, such a problem can be treated as problem with different material properties in a bar finite element volume.

Two main approaches for elastic-plastic linear hardening analysis of framed structures are found in the literature, namely the plastic hinge approach (Izzuddin and Elshai 1993a; Abbassia and Kassimali 1995) and the distributed plasticity approach (Izzuddin and Elshai 1993b; Meek and Loganathan 1990). In the first approach, the analysis is greatly simplified where the material plasticity is assumed to be concentrated at selected points, typically at the ends of finite elements. Such a concept has significant computation advantage, but analysis in most cases is suitable for preliminary approximate study as it is unable to deal with the spread of material plasticity. Moreover, plastic hinge approach is quite complex to adapt for hardening material case as it would be for elastic-perfectly plastic case.

In the second approach plastic strains are distributed along the element length. This case is still not widely investigated for frame structures. Usually hardening matrices are being introduced (Čyras 1986; Hongwu et al. 2003), but realization especially of bigger optimization problems is complicated. Standard mathematical software often can not solve nonlinear equations of such problems. Furthermore, in the second approach plastic strains distribution inside element section is evaluated only approximately. There are no such problems for truss or plate structures, since, in case of truss, plastic strain is developing at once in the whole element. In case of plane structures, plastic strains distribution problem can be solved by densification of finite elements mesh.

In most papers (Čyras et al. 2004; Kalanta 2007) related to finite elements under bending and compression or tension, strains are considered as rotation angles and axial displacements or plastic multipliers in plastic analysis case. But another treatment is also possible, which is of longitudinal rotational and linear strains. As it will be shown in this paper, the last-mentioned treatment is more convenient to solve physically nonlinear (or strain hardened) structures by incremental loading method, the problem becomes better realizable and understandable.
Thus, the main aim of this work is to describe and realize elastic-plastic frame structures analysis problem by finite element method, when plastic strains are treated as distributed in some volumes of frame elements.

1. Main assumptions and material hardening

There are such assumptions related to frame geometry, material, loading and analysis in this paper:
- cross-sections are not tapered along beams lengths;
- unloading phenomenon in plastic zones is evaluated;
- load varies slowly without dynamic effects;
- strains are small, i.e. equilibrium and geometrical equations are created for non-deformed structure schema;
- stability of structure is not evaluated;
- bending moments and axial forces are evaluated for both – elastic and plastic stages of material work;
- hardening is linear.

Assumption of linear hardening can be described by diagram (Fig 1) composed of two deformations fields – elastic and elastic-plastic.

![Stress-strain relation for linear hardening case](image)

2. Mathematical model and strains conversion

Most common mathematical models for analysis or optimization of linear hardened material structures imply so called hardening matrices (Чирас 1986; Hongwu et al. 2003). For instance in paper (Чирас 1986) such a static formulation of extreme energy principle is stated: of all statically admissible residual internal forces vectors, the one that sum of self-equilibrium internal forces and hardening elastic potentials is minimal is the true one.

This principle is represented by such mathematical model of nonlinear optimization problem:

\[
\begin{aligned}
&\frac{1}{2}S^T[D]S_r + \frac{1}{2}\lambda^T[H]\lambda \\
&\Rightarrow \min \\
&[A]S_r = 0 \\
&S_r - [\Phi]S_r + [\Phi]S_r + [H]\lambda \geq 0
\end{aligned}
\]

here analysis problem is expressed by elastic \(S_r\) and residual \(S_r\) internal forces vectors, \((S = S_r + S_s) - total internal forces vector); elastic properties of structure are described by symmetric flexibility matrix \([D]\); plastic properties of structure are described by matrix of yield conditions \([\Phi]\), hardening matrix \([H]\) and vector of limit internal forces \(S_0\); \([A]\) – equilibrium equations ratios matrix. Conditions equalities represent equilibrium of structure, conditions inequalities are yield conditions. Plastic multipliers \(\lambda\) and \(S_r\) are unknowns of this problem.

As it was mentioned earlier, there are problems of using hardening matrices for structures analysis or optimization. In case of plastic strains distribution concept hardening matrix involves additional unknowns – relative lengths of plastic strains zones in element (Чирас 1973). Therefore, mathematical model (3) contains not only nonlinear objective function, but nonlinear yield conditions too. In case of plastic hinge concept realization of mathematical model (3) is not complicated. But such an approach is based more on predictable structure behavior and results therefore can be treated only as preliminary and approximate.

An incremental loading method will be applied to solve such a problem. A standard mathematical model (Čyraš 1990; Merkevičiūtė and Atkocīnas 2006; Kalanta et al. 2009) composed of equilibrium, geometrical and physical equations can be used for one iteration cycle:

\[
\begin{bmatrix}
[A] & & \\
& & \\
[D] & & \\
\end{bmatrix}
\begin{bmatrix}
S \\
\theta \\
F
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix},
\]

where \(\theta\) - strains vector, \(u\) – displacements vector, \(F\) – forces vector. There are plenty literature (Kalanta and Grigusevičius 2004; Čyraš 1990) about fundamentals of
creation of these equations for two types of finite elements (Fig 2) so I will not describe it in detail.

In analysis problems of linear hardened structures is necessary to introduce plastic strains development concept. It can be evaluated as reduction of elasticity modulus $E$ in structure sections with plastic strains. To that end, a new matrix $[\varepsilon]$ is introduced, the members of which denote elasticity modules reductions of structure sections due to plastic strains development. Creation of this matrix is described in next chapter in detail.

Mathematical model (4) includes strain vector $\phi$ which is composed of rotation angles $\theta$ (in case of bending structures) and elongations $\Delta$ (if influence of axial forces is evaluated) $\phi_k = \{\varphi_{k1}, \varphi_{k2}, \Delta_k\}$. However, to use the longitudinal strains $\varepsilon$ (which are shown in Fig 1) is more convenient for analysis of physical nonlinear structures $\varepsilon_k = \{\varepsilon_{k,M1}, \varepsilon_{k,M2}, \varepsilon_{k,N,k}\}$, here $\varepsilon_{k,M1}, \varepsilon_{k,M2} -$ longitudinal strains caused by bending moments in upper or bottom layers of finite element sections; $\varepsilon_{k,N,k} -$ longitudinal strain caused by axial force in finite element. Sum of longitudinal strains caused by bending moment and axial force in one section $j$ $\varepsilon_j = \varepsilon_{j,M} + \varepsilon_{j,N,k}$ represent strain, which is shown in Fig 1. Conversion to strains $\varepsilon_k$ for two sections finite element under bending and tension or compression (Fig 2a) can be performed in such a way. The relation between internal forces $S_k$ and strains $\varepsilon_k$ in finite element $k$ (Fig 2a) is described as follows:

$$\begin{bmatrix}
\varphi_{k1} \\
\varphi_{k2} \\
\Delta_k
\end{bmatrix} =
\begin{bmatrix}
1/3E_kI_k & 1/6E_kI_k & 1/6E_kI_k \\
1/6E_kI_k & 1/3E_kI_k & 1/3E_kI_k \\
1/E_kA_k & 1/E_kA_k & 1/E_kA_k
\end{bmatrix}
\begin{bmatrix}
M_{k1} \\
M_{k2} \\
N_k
\end{bmatrix}
$$

(5)

or $\phi_k = [D]_k S_k$.

Here $I_k$ – cross-section inertia moment, $A_k$ – cross-section area. Meanwhile longitudinal strains $\varepsilon_k$ and internal forces $S_k$ are described by such a matrix formula (Čižas 1993):

$$
\varepsilon_k =
\begin{bmatrix}
\varepsilon_{k,M1} \\
\varepsilon_{k,M2} \\
\varepsilon_{k,N,k}
\end{bmatrix} =
\begin{bmatrix}
1/E_k W_k \\
1/E_k W_k \\
1/E_k W_k
\end{bmatrix}
\begin{bmatrix}
M_{k1} \\
M_{k2} \\
N_k
\end{bmatrix}
$$

(7)

or $\varepsilon_k = [D]_k S_k$ or $S_k = [D]_{\varepsilon,k}^{-1} \varepsilon_k$.

(8)

Here $W_k$ – cross-section resistance moment, $[D]_k$ – converse flexibility matrix of finite element. Physical equations (7) are more convenient (as compared with (5)) for analysis of structures with material nonlinearity since strains and internal forces are linearly related. Using (6) and (8) equations, longitudinal strains $\varepsilon_k$ and strains $\phi_k$ can be described by following relation:

$$\phi_k = [D]_k [D]_{\varepsilon,k}^{-1} \varepsilon_k = [Z] \varepsilon_k,$$

(9)

here $[Z]$ – strains conversion matrix. For two nodes finite element (Fig 2a) it will be as follows:

$$
[Z] =
\begin{bmatrix}
l_k W_k & l_k W_k & 6l_k \\
l_k W_k & l_k W_k & 6l_k \\
l_k W_k & l_k W_k & 6l_k
\end{bmatrix}
\begin{bmatrix}
l_k \\
l_k \\
l_k
\end{bmatrix}
= l_k W_k
\begin{bmatrix}
2 & 1 \\
3l_k & 6I_k \\
6l_k & 3I_k
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 2 \\
1 & 6l_k/W_k
\end{bmatrix}
$$

(10)

**Fig 2.** Generalized forces, nodal internal forces and generalized displacements of two finite elements types:

a) element under bending and compression or tension; b) element under bending and compression or tension with integral displacement $u_{i7}$.
Analogously, strains conversion matrix $[Z]_k$ is derived for finite element of second type (Fig 2b):

$$[Z]_k = \frac{I_k W_k}{15 I_k}$$

(11)

Conversed stiffness matrices of two types finite elements are:

- first type element

$$[K_z]_k = [D_z]_k^{-1} = \begin{bmatrix} E_k W_k & 0 & 0 \\ 0 & E_k W_k & 0 \\ 0 & 0 & E_k A_k \end{bmatrix}$$

(12)

- second type element

$$[K_e]_k = [D_e]_k^{-1} = \begin{bmatrix} E_k W_k & 0 & 0 \\ 0 & E_k W_k & 0 \\ 0 & 0 & E_k A_k \end{bmatrix}$$

(13)

Structures with material nonlinearity usually are solved by dividing external loads into increments, i.e. loading is being increased till its final value a certain interval of time. Thus, time parameter $t_F$ is introduced in the mathematical model (4), and the solving is performed by series of iterations evaluating loads increases $F_i$ in each of iterations (here $y_u$ – load multiplier of one iteration, $i$ – the number of iteration).

Applying all above mentioned modifications and equations (8), (9), (12), the mathematical model (4) takes a new form:

$$\begin{bmatrix} \phi \end{bmatrix} [K_e]_k [\varepsilon] - [S]_0 = 0$$

(14)

where $[\varepsilon] = \text{diag}([\varepsilon]), [K_e]_k = \text{diag}([K_e]_k)$ – quasi-diagonal matrices of all structure diagonally filled with the blocks of matrices $[Z]_k$ and $[K_z]_k$ respectively.

Usually plenty of load increments must be applied to solve structures with material nonlinearity, i.e. the smaller load increment – the better problem accuracy. Linear load optimization problem with strains constraints can be used to reduce the number of iterations. In this case maximum load increment multiplier $\max y_u$ is founded for uprisings of plastic strains at every section.

Uprising of plastic strains mathematically can be described by strains constraints $\varepsilon \leq \varepsilon_0 - \varepsilon_{tot,pr}$. Thus, now mathematical model (14) takes such a form: find

$$\begin{bmatrix} \max y_u \end{bmatrix}$$

when

$$\begin{bmatrix} [A] \varepsilon = y_u F \\
[A]^T \varepsilon - [Z]_k \varepsilon = 0 \\
\phi \begin{bmatrix} [K_e]_k \varepsilon - [S]_0 = 0 \\
\varepsilon \leq \varepsilon_0 - \varepsilon_{tot,pr} \end{bmatrix}$$

(15)

here $\varepsilon_{tot,pr}$ – known vector of total strains of structure sections at previous iteration; $\varepsilon_0$ – known vector of limit (yield) strains (limit strains values are limited only for sections, where total strains did not reach yield strain value at previous iteration, otherwise, yield strain is practically unlimited – $100000$), $\varepsilon$, $S$, $u$ – unknown vectors of strains, internal forces and displacement increments.

Additionally, the values of stresses at the top or bottom layers of the cross-sections $j$ have to be calculated:

$$\sigma_{M,j} = \frac{M_j}{W_k}$$

(16)

$$\sigma_{N,k} = \frac{N_k}{A_k}$$

(17)

$$\sigma_j = \sigma_{M,j} + \sigma_{N,k}$$

(18)

here $\sigma_{M,j}$ – maximum stress at $j$-th section caused by bending moment; $\sigma_{N,k}$ – stress at $k$-th finite element caused by axial force; $\sigma_j$ – total stress at $j$-th section caused by bending moment and axial force.

2. Evaluation of plastic strains distribution in the finite element

An influence of plasticity to structure work is suggested to evaluate by locating places and defining volumes of plastic strains (Fig 3), and depending on this, to reduce elasticity modules $E$ of corresponding sections in matrix $[K_e]_k$. These reductions are denoted by matrix of reduction ratios $\phi$, the range of which must be equal to matrix $[K_e]_k$ range, and non-zero members are placed in diagonal. If plastic strains is not developing in a certain section, corresponding member of matrix $\phi$ is equal to 1 (i.e. reduction of elasticity modulus is not required). If plastic strain is developing, the member value of matrix $\phi$ of corresponding section will be lesser than 1. These reduction ratios are calculated by such a relation:
Fig 3. Plastic strains (or stresses) distributions in the finite element section and along the element length:
a) plastic stresses at cross-section; b) plastic stresses distribution along the element; c) stresses distribution along the height of the section; d) distribution of stresses of the top layers; e) distribution of stresses of the bottom layers

\[ \Phi_i = \frac{E_{\text{mean},i}}{E} , \]  
here \( E_{\text{mean},i} \) – mean of elasticity modulus value at \( i \)-th side of finite element. \( E_{\text{mean},i} \) is calculated as follows:

\[ E_{\text{mean},i} = E - E_{\text{red},i} , \]  
here \( E_{\text{red},i} \) – reduction and of elasticity modulus. It is presumed, that the value of \( E_{\text{red},i} \) is proportionally equal to the following ratio:

- in case of bending moment evaluation:
\[ E_{\text{red},i} = r_i (E - E_h) \frac{V_{\text{pl},j}}{V_j} , \]  
where \( V_j \) – half of finite element volume at \( j \)-th section of finite element (third of volume for second type of element (Fig 2b)); \( V_k \) – volume of the finite element \( k \); \( V_{\text{pl},j} \) – plastic strains volume at \( j \)-th section of the finite element; \( V_{\text{pl},k} \) – plastic strains volume of the finite element \( k \); \( r_i \) – ratio of internal force (bending moment or axial force) evaluation to elasticity modulus reduction. For one finite element \( k \) (Fig 2a) the vector of these ratios is:
\[ r_k = (r_{M,j}, r_{M,j+1}, r_{N,k})^T \]  
for finite element of the second type (Fig 2b) – \( r_k = \left( r_{M,j}, r_{M,j+1}, r_{M,j+2}, r_{N,k} \right)^T \). It is presumed, that the ratios \( r_i \) are proportionally equal to the following relations (Fig 4):

- in case of bending moment evaluation:
\[ r_{M,j} = \frac{\sigma_{M,j}}{\sigma_{M,N,k}} \]  
\[ r_{M,j+1} = \frac{\sigma_{M,j+1}}{\sigma_{M,N,k}} \]  
here \( \sigma_{M,N,k} \) – absolute value of the mean stress in the finite element \( k \) caused by axial force (Fig 4b); \( \sigma_{m,M,k} \) – absolute value of the mean stress in the finite element \( k \) caused by bending moment; \( \sigma_{m,M,j} \) and \( \sigma_{m,M,j+1} \) – absolute values of the mean stresses in the \( j \)-th and \( (j+1) \)-th halves of the finite element \( k \) caused by bending moment. The mean stress \( \sigma_{m,N,k} \) is unvarying in any point of the element and therefore can be calculated by formula (17), meanwhile the mean stresses \( \sigma_{m,M,j} \), \( \sigma_{m,M,j+1} \) and \( \sigma_{m,M,k} \) vary along the section height and finite element length (Fig 4) and are calculated as follows (for the cross-section of I shape):

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Fig 4. Values of the mean stresses along the element length and cross-section height

\[ \sigma_{m,M,j} = \frac{A_{f,k}}{A_k} \left[ \sigma_{m,f,j} + A_{w,k} \sigma_{m,w,j} \right] \]  

(26)

\[ \sigma_{m,M,j+1} = \frac{A_{f,k}}{A_k} \left[ \sigma_{m,f,j+1} + A_{w,k} \sigma_{m,w,j+1} \right] \]  

(27)

\[ \sigma_{m,M,k} = \frac{A_{f,k}}{A_k} \left[ \sigma_{m,f,k} + A_{w,k} \sigma_{m,w,k} \right] \]  

(28)

Here \( \sigma_{m,f,j} \), \( \sigma_{m,w,j} \), \( \sigma_{m,f,j+1} \), \( \sigma_{m,w,j+1} \), \( \sigma_{m,f,k} \), and \( \sigma_{m,w,k} \) are absolute values of the mean stresses (Fig 4b) caused by bending moments along the height of the flanges and web in the \( j \)-th and \( (j+1) \)-th halves of the finite element \( k \) and at the whole finite element \( k \); \( A_{f,k} \) – area of the flanges; \( A_{w,k} \) – area of the web. For example, the mean stresses \( \sigma_{m,f,j} \) and \( \sigma_{m,w,j} \) are calculated as follows:

\[ \sigma_{m,f,j} = \sigma_{m,M,j} \left[ 1 - \frac{t_j}{h} \right] \]  

(29)

\[ \sigma_{m,w,j} = \sigma_{m,M,j} \left[ 0.5 - \frac{t_j}{h} \right] \]  

(30)

here \( \sigma_{m,M,j} \) – absolute value of the mean stress caused by bending moments along the length of the \( j \)-th half of the finite element \( k \). For the case of stress distribution shown in Fig 4a the mean stress \( \sigma_{m,M,j} \) is calculated:

\[ \sigma_{m,M,j} = \frac{\sigma_{M,j} l_j + \sigma_{M,mid} l_{j,mid}}{2l_j + 2l_{j,mid}} \]  

(31)

Note that the distribution of stresses along the cross-section height in Fig 4b is not nonlinear and it is presumed that the error due to this linearization is very small.

Thus, finally the relation (19) is expressed as follows:

- in case of bending moment evaluation:

\[ \phi_\mu = 1 - \frac{V_{pl,j}}{V_j} + \alpha \gamma \frac{V_{pl,k}}{V_k} \]  

(32)

- in case of axial force evaluation:

\[ \phi_\mu = 1 - \frac{V_{pl,j}}{V_j} + \alpha \gamma \frac{V_{pl,k}}{V_k} \]  

(33)

The plastic strains volumes \( V_{pl,j} \) and \( V_{pl,k} \) have to be calculated separately analyzing every section of a structure since it depends on stress value, cross-section shape and plastic strains distribution along the element length. However, a set of formulas for all cases can be created. For example, for case of I shape section in Fig 3, plastic strain volumes are calculated as follows:

\[ V_{pl,j} = 0.5A_{pl,j}\left( t_{pl,j,t} l_{pl,j,t} + 0.5A_{pl,j,b} l_{pl,j,b} \right) + 0.5A_{pl,j,t} l_{pl,j,t} + 0.5A_{pl,j,b} l_{pl,j,b} = \]

\[ = 0.5\left( t_{j} + t_u \left( h_{pl,j,t} - t_{j} \right) \right) l_{pl,j,t} + 0.5\left( t_{j} + t_u \left( h_{pl,j,b} - t_{j} \right) \right) l_{pl,j,b} \]  

(34)

\[ V_{pl,k} = V_{pl,j} + V_{pl,j+1} \]  

(35)
here $A_{pl,j,t}$, $A_{pl,j,b}$ – plastic strains fields areas at the top and bottom of the section $j$; $l_{pl,j,t}$, $l_{pl,j,b}$ – lengths of plastic zones along the element $k$ at the top and bottom of the section $j$. Plastic zone heights $h_{pl,j,t}$, $h_{pl,j,b}$ in the element section (Fig 3c) are calculated as following (Jaras and Kačianauskas 1992):

$$h_{pl,j,t} = 0.5a \left(1 - \frac{\alpha \sigma_0}{\sigma_{M,j,t} + \sigma_{N,k} + \alpha \sigma_0}\right),$$  \hspace{1cm} (36)

$$h_{pl,j,b} = 0.5a \left(1 - \frac{\alpha \sigma_0}{\sigma_{M,j,b} + \sigma_{N,k} + \alpha \sigma_0}\right),$$  \hspace{1cm} (37)

here $\sigma_{M,j,t}$, $\sigma_{N,k}$ – maximum values of stresses caused by bending moment at $j$-th section and by axial force at $k$-th element.

The plastic zone lengths $l_{pl,j,t}$, $l_{pl,j,b}$ along the element at the $j$-th section (Fig 3b, 3d, 3e) can be calculated by creating proportions of stresses at element ends:

$$l_{pl,j,t} = \frac{\sigma_{M,j,t} - \sigma_{0,k}}{\sigma_{M,j,t} + \sigma_{N,k} + \alpha \sigma_0} l_k + \frac{\sigma_{N,k}}{\sigma_{M,j,t} + \sigma_{N,k} + \alpha \sigma_0} l_k,$$  \hspace{1cm} (38)

$$l_{pl,j,b} = \frac{\sigma_{M,j,b} - \sigma_{0,k}}{\sigma_{M,j,b} + \sigma_{N,k} + \alpha \sigma_0} l_k - \frac{\sigma_{N,k}}{\sigma_{M,j,b} + \sigma_{N,k} + \alpha \sigma_0} l_k.$$  \hspace{1cm} (39)

The relations (38), (39) are proper when $|\sigma_{M,j}| \geq \sigma_{0,k}$.

3. Solving algorithm of the mathematical model

An iterative solving algorithm based on the modified Newton-Raphson technique (He 2004; Ghali et al. 2009) will be applied to analyze structure with material non-linearity. In practical use, the load is introduced in stages, with convergence achieved in each stage (Fig 5). Solving algorithm is completed in seven steps:


Step 2. Create matrix $[\varphi]$. For the first iteration, this matrix is unit, as structure has only elastic strains. Define yield strain vector $\varphi_0$. For the first iteration, all strains must be constrained.

Step 3. Solve linear force optimization problem with strains constraints (15). Total strains are evaluated as sum of angular and linear strains

$$\varepsilon_{tot,j} = \sum_{n=1}^{N} \left(\varepsilon_{M,j,n} + \varepsilon_{N,k,n}\right).$$

Total displacements and internal forces are calculated in similar way

$$\delta_{tot,j} = \sum_{n=1}^{N} \delta_{n,m} + \sum_{n=1}^{N} u_{m,n}$$

(here $m$ – number of possible displacements).

Step 4. Corrective iteration ($B_1$, $B_2$, $B_3$, $C_1$, $C_2$ and $C_3$ points in Fig 4). Recalculate members of matrix $[\varphi]$ applying formulas (19)–(39). If values changed more than by 1 %, return to step 3 and correct load multiplier $y_u$ value.

Step 5. Main iteration ($A$, $B$, $C$ and $D$ points in Fig 4). If multiplier $y_u < 1$, then unrestrain corresponding members of vector $\varphi_0$ and return to step 3. If values of matrix $[\varphi]$ changed less than by 1 % and load multiplier $y_u \geq 1$, then accept $y_u = 1$ and go to step 6.

Step 6. Final iteration ($D_1$ and $D_2$ points in Fig 4). Solve the mathematical model (14) to correct final results. Recalculate members of matrix $[\varphi]$ applying formulas (19)–(39).

Step 7. If values of matrix $[\varphi]$ changed more than by 1 %, then return to step 6, otherwise solution is finished.

4. Numerical example

Three elements steel frame subjected by concentrated and distributed loads (Fig 6) was analyzed, MAPLE mathematical software was used. Steel properties: yield strength – 230MPa; elastic modulus – 205GPa; hardening modulus – 15GPa; yield strain – $\varepsilon_0 = 230 \cdot 10^{-3}/205 \cdot 10^6 = 0.00112195$. Geometrical characteristics: inertia modules $I_{x,3} = 171cm^4$, $I_2 = 573cm^4$; resistance moments $W_{x,3} = 34.2cm^3$, $W_2 = 81.9cm^3$; areas $A_{x,3} = 10.6cm^2$, $A_2 = 18.2cm^2$. Initial limit bending moments: $M_{01} = 7.866kNm$, $M_{02} = 18.837kNm$, $M_{03} = 7.866kNm$. Strength and stiffness requirements are not considered.

Only nine iterations were performed to solve this problem. Strains in three sections reached limit strains values. Table 1 shows the variation of strains, stresses and reductions ratios of elasticity modules in iteration process. Distributions of bending moments, axial forces and plastic strains places are shown in Fig 7 and 8.
### Table 1. Main results of numerical example

| Iteration | Load multiplier $y_i$ | Section | Strain, $\varepsilon = |\varepsilon_M| + |\varepsilon_N|$ | Limit strain, $\varepsilon_l$ | Stress, $\sigma = |\sigma_M| + |\sigma_N|$, kPa | Remarks (reduced matrix $[\phi]$ members after iteration) |
|-----------|------------------------|---------|-------------------|-------------------|-------------------|--------------------------------------------------|
| 1         | 0.467375               | 1       | 11.219            | 11.219            | 1                 | 230.00                                           | all matrix members in diagonal are equal to 1 |
|           |                        |         |                   |                   |                   |                                                  |                                                   |
| 2         | 0.558425               | 1       | 15.939            | 100000            | 1                 | 237.08                                           | $\phi_u = 0.903$, $\phi_{33} = 0.999$ |
|           |                        |         |                   |                   |                   |                                                  |                                                   |
| 3         | 0.580425               | 1       | 15.740            | 100000            | 1                 | 236.78                                           | $\phi_u = 0.913$, $\phi_{33} = 0.999$ |
|           |                        |         |                   |                   |                   |                                                  |                                                   |
| 4         | 0.58090                | 1       | 15.758            | 100000            | 1                 | 236.81                                           | $\phi_u = 0.912$, $\phi_{33} = 0.999$ |
|           |                        |         |                   |                   |                   |                                                  |                                                   |
| 5         | 0.7467                 | 1       | 30.822            | 100000            | 1                 | 259.40                                           | $\phi_u = 0.827$, $\phi_{22} = 0.913$, $\phi_{33} = 0.997$ |
|           |                        |         |                   |                   |                   |                                                  |                                                   |
| 6         | 0.73805                | 1       | 30.036            | 100000            | 1                 | 258.22                                           | $\phi_u = 0.833$, $\phi_{22} = 0.917$, $\phi_{33} = 0.997$ |
|           |                        |         |                   |                   |                   |                                                  |                                                   |
| 7         | 1.000                  | 1       | 112.69            | 100000            | 1                 | 382.21                                           | $\phi_u = 0.753$, $\phi_{22} = 0.830$, $\phi_{33} = 0.994$, $\phi_{88} = 0.787$, $\phi_{1010} = 0.990$ |
|           |                        |         |                   |                   |                   |                                                  |                                                   |
| 8         | 1.000                  | 1       | 112.69            | 100000            | 1                 | 382.21                                           | $\phi_u = 0.752$ (1 section), $\phi_{22} = 0.825$ (2 section), $\phi_{33} = 0.994$ (1 element), $\phi_{88} = 0.804$ (7 section), $\phi_{1010} = 0.991$ |
|           |                        |         |                   |                   |                   |                                                  |                                                   |
| 9         | 1.000                  | 1       | 112.69            | 100000            | 1                 | 382.21                                           | $\phi_u = 0.752$ (1 section) |
|           |                        |         |                   |                   |                   |                                                  |                                                   |

**Fig 6.** Frame design schema

**Fig 7.** Bending moments distribution and plastic strains places
Fig 8. Axial forces distribution in the frame

Conclusions

Volumetric plasticity concept for linear hardening steel frames analysis is suggested in this paper. This concept allowed evaluating plastic strains distribution inside element section and along element length. Conversion of strains and reduction of elasticity modules (described in this work) made analysis problem with material nonlinearity well-understandable and easily realizable, herewith data of such a problem can be easily arranged for structures analysis or optimization software creation.

Modified Newton-Raphson method technique was adapted to the problem analysis and applied for numerical example. Only nine iterations were required to solve the frame of three elements. The simplest analysis technique known as the incremental method usually requires not less than one hundred iterations (if high accuracy is needed). Therefore, it can be proposed, that especially for small problems, modified Newton-Raphson method is much more practical as it requires less computation time.

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