ASSESSMENT OF POTENTIAL MECHANICAL DAMAGE
TO TANKS OF FLAMMABLE LIQUIDS

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Abstract. Accidental (abnormal) actions are among the main causes of structural failures. In this paper an estimation of probability of structural failure of tank with flammable liquid due to an accidental action is considered. Two sources of evidence are applied to this estimation: a small-size statistical sample and a fragility function. This function expresses aleatory and epistemic uncertainties related to the potential failure. The estimation of the failure probability is based on Bayesian reasoning. Bayesian prior and posterior distributions are applied to express the epistemic uncertainty in the failure probability. The prior distribution is developed by propagating epistemic uncertainty inherent in the fragility function and, if necessary, values of demand variables. The posterior distribution is estimated by carrying out Bayesian updating with uncertain (imprecise) data. Such updating is considered a sort of uncertainty propagation. The uncertain data is expressed by a set of continuous epistemic probability distributions of fragility function values. The distributions are generated by inserting elements of the small-size sample into the fragility function which is uncertain in the epistemic sense. The Bayesian updating with the new data represented by the set of continuous distributions is carried out by discretizing these distributions. The discretization yields a new sample which is entered into the Bayes theorem through likelihood function. The sample created by the discretization consists of fragility function values which have equal epistemic weights. The proposed scheme of discretization is considered an alternative to a posterior averaging approach. This approach is suitable for Bayesian updating with uncertain data; however, it is applicable to the case where data uncertainty is modeled by discrete distributions of epistemic uncertainty. Several aspects of numerical implementation of the proposed discretization approach and subsequent updating are discussed and illustrated by an example.

Keywords: Failure probability, Fragility, Tank explosion, Industrial accidents, Bayesian updating, Structural failure.

Introduction

Abnormal service conditions remain among the main causes of structural failures. In Eurocodes, these conditions are called the accidental situations; actions induced during an accidental situation are termed the accidental actions (ENV 1991-2-7:1998). An occurrence of an accidental situation is usually a highly random event of short duration. It can be caused (triggered off) by component failure, human error, man-made accident, extreme natural phenomenon. Accidental situations are inherently a natural subject of the quantitative risk assessment (QRA) (Vaidogas 2005, 2006, 2007a; Vaidogas and Juocevicius 2008b). The risk posed by accidental situations can be systematically quantified by applying Bayesian approaches to QRA (Aven 2009; Aven and Pörn 1998; Singpurwalla 2006) A probability of failure of a structure subjected to an accidental situation can be component of such risk (Vaidogas 2007a, 2007b; Vaidogas and Juocevicius 2008a).

Major accidents caused by explosions and fires due to leakage of flammable liquids and/or formation of ignitable air-vapour clouds in hazardous industries are not a rare phenomena, e.g. storage tank explosion and fire in Glenpol 2003 (NTSB 2003) (Fig 1); explosion and fire in Barton’s Solvents facility 2007 (CSB 2007); refinery fire at Feyzin 1966 (IChemE 1987); Explosion in Rafnes plant, Norway 1988 (Khan and Abbasi 1999) etc. The storage tank at ConocoPhillips Company’s Glenpool South tank farm contained about 7600 barrels of diesel at the time of explosion. The resulting fire burned for 21 hours and damaged two other tanks in the area. The monetary losses of this accident, including emergency response, environmental remediation, evacuation, lost product, property damage, and claims, was about 2,35 mln. USD. Major accidents often involve hazardous releases of large amount of energy and severe damage to the structural components of industrial infrastructure (Vaidogas and Juocevicius 2008a). In many cases, structural components can be ‘key players’ in sustaining haz-
ardous phenomena and mitigating consequences of major accidents.

The aforementioned explosion and fire at the Barton Solvents distribution facility in Valley Centre, Kansas, destroyed the tank farm on July 17, 2007, at about 9 a.m. The incident triggered an evacuation of Valley Centre (approximately 6000 residents), caused injuries to 11 residents and significantly interrupted the Barton’s business. The fire in the facility was caused by an explosion of air-vapour mixture inside the storage tank with non-conductive flammable liquid due to failure of level-measuring float link. Event tree path of this accident is presented in Fig 2. The approach to the assessment of possible mechanical damage to tanks with flammable liquids, say the estimation of probabilities of structural failures, is presented in this paper.

The probability of structural failure due to an accidental situation is often estimated by decomposing the problem into two tasks: (a) predicting characteristics of an accidental action which can be induced during the situation and (b) developing a fragility function for the structure which can be subjected to the action. Such decomposition is widely used, for instance, in the earthquake risk assessment (Der Kiureghian 1999; Ellingwood 2001; Sasani et al. 2002; Lee and Rosowsky 2006; Li and Ellingwood 2007) and extreme wind risk assessment (Ravindra 1995; Ellingwood 2006). The tasks (a) and (b) may include the need to deal with sparse and imprecise evidence related to both accidental action and potential failure due to this action.

A relatively large number of methods are applied to estimating probabilities using sparse and imprecise evidence. These methods are based either on fuzzy logic, probability theory, possibility theory, or evidence theory (Der Kiureghian 1999; Sasani et al. 2002; Fetz and Oberguggenberger 2004; Tonon 2004; Hall and Lawry 2004; Baudrit et al. 2008). As far as we known, a comprehensive and comparative state-of-the-art review of the aforementioned methods is not available. Therefore it is difficult to say which of the above theories is best suited to expressing uncertainty related to fragility function values and estimating an imprecise probability of failure due to an accidental action. The probabilistic methods based on Bayesian reasoning define the failure probability as a measure of aleatory (irreducible) uncertainty; the imprecision related to the true albeit unknown value of this probability is called the epistemic (reducible) uncertainty (Aven 2009; Aven and Pörn 1998; Singpurwalla 2006). Unlike the aforementioned non-probabilistic approaches, the Bayesian approach allows to estimate the failure probability by a relatively simple and consistent propagation of aleatory and epistemic uncertainties (Vaidogas 2005, 2007a, 2007b; Vaidogas and Juocevicius 2009a). Results of this estimation can be embedded with relative ease in the higher-level QRA which considers, say, a domino process involving the structural failure under analysis and yields the final risk profile (Vaidogas and Juocevicius 2008a; Salzano and Cozzani 2005).

In case where a fragility function is developed in the Bayesian format, the failure probability can be estimated by carrying out a Bayesian inference with uncertain (imprecise) data (Vaidogas 2007b; Vaidogas and Juocevicius 2009a). Such data is generated in the course of the failure probability estimation. The data is represented by continuous epistemic probability distributions of fragility function values. These distributions express the epistemic uncertainty inherent in the fragility function.

**Fig 1.** Explosion and fire at the ConocoPhillips Company’s Glenpool South tank farm

**Fig 2.** Event tree path that resulted the destruction of Barton’s Solvents distribution facility in Valley Center, Kansas

In the context of assessing the failure probability of a tank system, a fragility function is used to quantify the probability of failure given a specific level of input energy. The fragility function is typically expressed as a probability density function or a cumulative distribution function, which provides a measure of the likelihood of failure for a given level of input energy. The fragility function is developed based on historical data, experimental results, and expert judgment, and it is used in conjunction with the event tree to assess the potential consequences of a failure event.
The probability of damage

Let the random event $\mathcal{F}$ stand for a potential failure of a structure due to an accidental situation represented by the random event $\mathcal{A}\mathcal{F}$. The conditional probability of $\mathcal{F}$ can be expressed in the form of a mean value (Vaidogas 2005; Vaidogas 2007b; Vaidogas and Juocevicius 2009a):

$$
\mu = P(\mathcal{F} | \mathcal{A}\mathcal{F}) = \frac{\int P(\mathcal{F} | y) dF_y(y) - E_y(P(\mathcal{F} | Y))}{\text{all } y}
$$

where $Y$ is the random vector representing characteristics of the accidental action generated during the occurrence of $\mathcal{A}\mathcal{F}$, and $F_y(y)$ are the value of $Y$ and its distribution function (d.f.), respectively; $P(\mathcal{F} | Y)$ is the fragility function (f.f.) with the demand variables $y$; $E_y(\cdot)$ denotes the mean value with respect to $Y$; and $P(\mathcal{F} | Y)$ denotes a function of the random vector $Y$.

The function $P(\mathcal{F} | y)$ expresses the aleatory uncertainty in occurrence of $\mathcal{F}$ given an accidental action with characteristics $y$. However, values of $P(\mathcal{F} | y)$ can be uncertain in the epistemic sense. Several different approaches were proposed to model the epistemic uncertainty in $P(\mathcal{F} | y)$ (Der Kiureghian 1999; Ellingwood 2001; Sasani et al. 2002; Ravindra 1995). A systematic review of these approaches is not available at present. However, the most consistent approach seems to be developing $P(\mathcal{F} | y)$ by means of Bayesian parameter estimation (Der Kiureghian 1999; Sasani et al. 2002). The epistemic uncertainty in $P(\mathcal{F} | y)$ is expressed through the Bayesian limit state function $g(Z, y | \Theta)$. In this function, $Z$ is the vector describing the structure exposed to an accidental action and $\Theta$ denotes the vector of model parameters. With a fixed (crisp) $\Theta$, $P(\mathcal{F} | y)$ expresses aleatory uncertainty and is defined as the probability $P(g(Z, y | \Theta) \leq 0)$, where $Z$ is the random vector quantifying the aleatory uncertainty in $Z$.

$$
F(y | \Theta) = P(\mathcal{F} | y, \Theta) = P(g(Z, y | \Theta) \leq 0)
$$

The following consideration seeks to answer the question, how to estimate $P(\mathcal{F} | y)$ by applying two kinds of evidence about the accidental situation under analysis: (a) the f.f. $F(y | \Theta)$ and (b) a small-size statistical sample $y$ consisting of experimental observations of $y$:

$$
y = \{y_1, y_2, ..., y_j, ..., y_n\}
$$

where $y_j$ is the value of $y$ recorded in the $j$th experiment. The case is considered where the size $n$ of $y$ is too small to fit the d.f. $F_y(y)$ in the standard statistical way. This case is realistic one because experiments imitating an accidental situation can be too expensive to obtain a large-size $y$.

Estimating the damage probability

The prior $\pi(\mu)$ of the mean value $\mu$ defined by Eq (1) can be specified by utilizing knowledge about the accidental situation under study (Vaidogas 2007b; Vaidogas and Juocevicius 2009a). Such knowledge, more or less relevant to the situation, can often be represented by the mathematical model expressed by the vector-function

$$
y = \nu(X | \xi)
$$

where $x$ is the argument vector which represents evidence allowing to predict the action characteristics $y$; $\xi$ is vector of those parameters of $\nu(\cdot)$ which are uncertain in the epistemic sense. Values of $x$ may be uncertain in the aleatory sense and this uncertainty can be modeled by a random vector $X$ with an aleatory d.f. $F_x(x)$. Epistemic uncertainties related to $\xi$ will be expressed by a random vector $\xi$ with a d.f. $F_\xi(\xi)$.

Replacing $Y$ in the function $P(\mathcal{F} | Y)$ by the random function $\nu(X | \xi)$ and averaging out the aleatory uncertainty expressed by $X$ yield the epistemic r.v.

$$
M = E_x(F(\nu(X | \xi) | \Theta)) = \int F(\nu(x | \xi) | \Theta) dF_x(x)
$$

A value of $M$ is the failure probability corresponding to given values $\xi$ and $\Theta$ of $\xi$ and $\Theta$.

The prior knowledge expressed by the model $\nu(\cdot)$ may be only partially relevant to the potential accidental situation. The source of the partial irrelevance may lie in structure of $\nu(\cdot)$ and/or data used to fit the d.f. $F_x(x)$ and estimate the parameters $\xi$. The new data necessary for estimating $\mu$ can be derived from the sample $y$. This sample will be suitable for estimating if it is statistically representative and relevant to the accidental situation.

The $j$th element $y_j$ of the sample $y$ generates an epistemic r.v.

$$
P_j = F(y_j | \Theta)
$$

which can be treated as uncertain datum with its own p.d.f. $f_j(p)$ and d.f. $F_j(p)$ (Figs 3 and 4). Consequently, the f.f. $F(y | \Theta)$ requires to update $\pi(\mu)$ using a set of $n$ uncertain data $\{P_1, P_2, ..., P_j, ..., P_n\}$.

If necessary, the r.v.s $P_j$ can express the epistemic uncertainty related to the observations $y_j$. Such uncertainty may be related to values of the demand variables, $y_j$, which are determined in the post-mortem investigations of previous accidents (occurrences of $\mathcal{A}\mathcal{F}$). Alternatively, the experimental observation $y_j$ may not fit completely in the potential accidental situation under study. Therefore expert judgment may be necessary to correct (adjust) some or all of $y_j$. A subjective adjustment by experts may also be necessary in the case where the values $y_j$ are obtained by a numerical simulation of accidental situation, for instance, a CFD simulation of fire or explosion. The uncertainty in $y_j$ can be modeled by an epistemic random vector $Y_j$ with the p.d.f. $f_j(y)$. Then the epistemic r.v.
posterior averaging can be applied by discretizing the distributions of \( P_j \) in the traditional way. However, these distributions can be discretized and the prior \( \pi(\mu) \) updated without using the posterior averaging. The heuristic principle of such a discretization is that this process should yield \( m \) values \( p_k \) of \( P_j \) and these values should have equal epistemic weights \( w_k = 1/m \) (\( k = 1, 2, \ldots, m \)). The equal weights \( w_k \) assure that none of the values \( p_k \) will be preferred to others. The equal weights \( w_k \) are an analogy with the equal attitude towards elements of a sample collected by following a standard probability sampling scheme (e.g. Barnet 1991: 106).

The suggested discretization is illustrated in Fig 3b,c. The values \( p_k \) can be calculated by

\[
p_k = F_j^{-1}(k/(m+1)) \quad (k = 1, 2, \ldots, m)
\]

where \( F_j^{-1}(\cdot) \) is the inverse d.f. of \( P_j \). The non-uniformly arranged values \( p_k \) can be interpreted as ones of a r.v. with the probability masses \( w_k \) equal to \( 1/(m+1) \) (Fig 3b). The discretization leads to a loss of the upper tail area \( 1 - \Phi_j^m \) (Fig 3a), and so \( w_k \) do not strictly satisfy the condition \( \sum_k w_k = 1 \). However, this discrepancy will decrease when the number \( m \) increases.

After the transformation (8) is applied to all \( n \) elements of the sample \( y \), a new sample consisting of \( n \times m \) elements is obtained:

\[
p = \{ (p_{jk}, k = 1, 2, \ldots, m), j = 1, 2, \ldots, n \}
\]

When the same number \( m \) is applied to discretize each \( P_j \), all elements of \( p \) will have equal epistemic weights approximately equal to \( 1/m \). Then the sample \( p \) defined by Eq (9) can be applied to updating the prior \( \pi(\mu) \).

The usual Bayesian posterior \( \pi(\mu | data) \) is proportional to the product \( \pi(\mu) \times L(data | \mu) \), where \( L(data | \mu) \) is the likelihood function and “data” is represented by \( p \). The posterior \( \pi(\mu | data) \) can be replaced by an estimated one (Vaidogas 2007b; Vaidogas and Juocevicius 2009a):

\[
\hat{\pi}(\mu | data) \propto \pi(\mu) \hat{L}_B(data | \mu)
\]

where \( \hat{L}_B(data | \mu) \) is an estimate of \( L(data | \mu) \) based on bootstrap estimation of the density of the pivotal quantity \( \hat{\mu}_{ncm} - \hat{M} : \hat{\mu}_{ncm} \) is the mean value of the sample \( p \).
Example

The failure probability $P(F \mid \mathcal{F})$ is to be estimated for an accidental situation, which can be caused by an accidental explosion within a 150x200 m² zone of a plant processing industrial explosives (Fig 5). The failure $F$ consists in a loss of containment of a steel tank built outside the zone due to the action of the blast wave generated by the explosion. The prior knowledge is expressed by the model

$$y = v(x) = v'(z_y(x_1, x_2)) =$$

$$v\left(\left\{ \frac{0.43x_1^{2/3}}{r(x_2, x_3)} + \frac{1.4x_1}{r^2(x_2, x_3)} + \frac{1.4x_1}{r^3(x_2, x_3)} \right\} \right)$$

where $y$ is the peak positive overpressure of the blast wave reflected by the tank; $r(x_1, x_2)$ is the standoff of the explosion (Fig 5); $v'$ is the deterministic function used to transform the incident peak overpressure into the reflected one (Kotlerovskij et al. 1995); $z_y$ is the dimensionless factor used to adjust the standard trinitrotoluol model $y(x_1, x_2)$ to the explosive under analysis.

The aleatory uncertainty is related to arguments of $v(x \mid z_y)$ and expressed by the random vector $X = (X_1, X_2, X_3)^T$. Its components are the normally distributed mass of explosive, $X_1 \sim \text{N}(30 \, \text{kg}, 3 \, \text{kg})$, and the uniformly distributed coordinates of explosion centre, $X_2 \sim \text{U}(0 \, \text{m}, 150 \, \text{m})$ and $X_3 \sim \text{U}(0 \, \text{m}, 200 \, \text{m})$. The epistemic uncertainty is introduced into the prior knowledge by assuming that the adjustment factor $z_y$ is uncertain in the epistemic sense. This uncertainty is modeled by a lognormal r.v. $\mathcal{Z} \sim \text{L}(0.17975; 0.11957)$.

The vulnerability of the tank to the explosion is expressed by an f.f. $F(y \mid \Theta)$. The f.f. $F(y \mid \Theta)$ is a hypothetical one and represented by a d.f. of a univariate normal distribution, $F(y \mid \Theta_1, \Theta_2)$, with uncertain mean $\Theta_1$ and uncertain variance $\Theta_2$. These parameters are assumed to be independent and distributed as indicated in Table 1.

<table>
<thead>
<tr>
<th>Parameter of f.f.</th>
<th>Type of prior</th>
<th>Parameters of the prior distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Theta_1$</td>
<td>Normal</td>
<td>7 kPa (mean); 0.77 kPa (sd. dev.)</td>
</tr>
<tr>
<td>$\Theta_2^{-1}$</td>
<td>Gamma</td>
<td>18 (shape); 14,962 (kPa)−1 (scale), 1,1362 (kPa)−2 (mode)</td>
</tr>
</tbody>
</table>

* According to recommendations of Congdon (2000: 19).

### Table 1. Prior distributions of the fragility function parameters $\Theta_1$ and $\Theta_2$.

<table>
<thead>
<tr>
<th>Type of prior</th>
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<tr>
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</tr>
</tbody>
</table>

### Table 3. Descriptive measures of the samples $p$, $p'$, and $p''$ used in the first and second examples.

<table>
<thead>
<tr>
<th>Sample size</th>
<th>Mean</th>
<th>Std.dev.</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Minimum</th>
<th>Maximum</th>
<th>10th prc.</th>
<th>90th prc.</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>0.02048</td>
<td>5.69210-8</td>
<td>1.78</td>
<td>2.02</td>
<td>5.692–10–8</td>
<td>6.402–10–3</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>450</td>
<td>0.013234</td>
<td>0.038145</td>
<td>5.2357</td>
<td>34.331</td>
<td>5.60–10–14</td>
<td>0.3724</td>
<td>2.27–10–7</td>
<td>0.03406</td>
</tr>
<tr>
<td>900</td>
<td>0.013261</td>
<td>0.039008</td>
<td>5.5544</td>
<td>39.680</td>
<td>3.89–10–15</td>
<td>0.4356</td>
<td>1.94–10–7</td>
<td>0.03414</td>
</tr>
<tr>
<td>900 000</td>
<td>0.013197</td>
<td>0.040145</td>
<td>6.3105</td>
<td>55.820</td>
<td>0.0</td>
<td>0.9590</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

### Table 4. Descriptive measures of the simulated samples $p_j$ obtained with $n = 100$ 000 and computed for the elements $y_j$ of the initial sample $y$.

<table>
<thead>
<tr>
<th>$j$</th>
<th>$y_j$ (kPa)</th>
<th>Mean</th>
<th>St.dev.</th>
<th>Kurt.</th>
<th>Skewness</th>
<th>Minimum</th>
<th>Maximum</th>
<th>10th prc.</th>
<th>90th prc.</th>
</tr>
</thead>
</table>
The sample element \(\xi_0,30\) at \(\mu\) sample (Fig 6a). The transformation distribution \(Ga(0,1557; 17,49)\) was fitted to the latter is an estimate of the mean value. It implies that the prior \(\pi(\mu)\) can be specified by fitting it to the sample \(\{\mu_1, \mu_2, \ldots, \mu_i, \ldots, \mu_{n_1}\}\). The sample element \(\mu_i\) is an estimate of the mean value \(E_X(\nu(X | \xi_i | \theta_j))\) at the given values \(\xi_i\) and \(\theta_j = (\theta_{i1}, \theta_{i2})^T\) (Eq (5)). The values \(\xi_i\) were sampled by means of a stochastic (Monte Carlo) simulation from \(L(0,17975; 0,11957)\). The values \(\theta_j\) were sampled from the distributions given in Table 1. The sample size \(n_1\) was assumed to be equal to 1000.

The sample of \(\mu_i\)s can be used to fit the prior \(\pi(\mu)\) which will express the epistemic uncertainty in \(P(\mathcal{F} | \mathcal{A} | \mathcal{F})\). It was problematic to fit a widely known univariate probability distribution to the sample \(\{\mu_1, \mu_2, \ldots, \mu_i, \ldots, \mu_{1000}\}\). Therefore this was transformed into the sample \(\{-\ln\mu_1, -\ln\mu_2, \ldots, -\ln\mu_{1000}\}\) and a gamma distribution \(Ga(0,1557; 17,49)\) was fitted to the latter sample (Fig 6a). The transformation \(\psi = -\ln\mu\) was chosen intuitively. It implies that the prior \(\pi(\mu)\) can be obtained from the p.d.f. \(f(\psi)\) of the r.v. \(\psi \sim Ga(\alpha = 0,1557; \beta = 17,49)\).

The new information used for updating was represented by the sample \(\mathbf{p}'\) obtained by clustering the nine samples \(\mathbf{p}_j, j = 1, 2, \ldots, 9\). The sample \(\mathbf{p}_j\) was computed by transforming the corresponding sample \(\mathbf{p}\). The linear transformation was not applied because it produced negative elements \(\mathbf{p}'_j\) of \(\mathbf{p}_j\) in all nine cases. The sample \(\mathbf{p}_j\) is a result of discretizing the r.v. \(P_j\) with the d.f. \(F_j(p)\) into a set of \(m\) quantiles \(p_k\) defined by Eq (8). As the d.f. \(F_j(p)\) is not known in the present case, the values

**Fig 6.** Graphs illustrating the choice of the prior \(\pi(\mu)\): (a) histogram of the sample \(\{-\ln\mu_1, -\ln\mu_2, \ldots, -\ln\mu_{1000}\}\) and the gamma density \(Ga(0,1557; 17,49)\) fitted to this sample; (b) transformed gamma prior \(\pi(\mu)\sim Ga(-0,1557; 17,49)\) with the mean of 0,08 and the coefficient of variation equal to 61,1 %

**Table 2.** New data \(y\) (experimental records of the overpressure \(y_j\)) and corresponding sample of f.f. values, \(p\)

<table>
<thead>
<tr>
<th>(j)</th>
<th>Charge (kg)</th>
<th>Standoff (m)</th>
<th>Reflected pressure (y_j) (kPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>27,0</td>
<td>117</td>
<td>3,767</td>
</tr>
<tr>
<td>2</td>
<td>26,9</td>
<td>142</td>
<td>4,276</td>
</tr>
<tr>
<td>3</td>
<td>28,2</td>
<td>132</td>
<td>4,160</td>
</tr>
<tr>
<td>4</td>
<td>31,5</td>
<td>125</td>
<td>3,944</td>
</tr>
<tr>
<td>5</td>
<td>29,3</td>
<td>92</td>
<td>4,916</td>
</tr>
<tr>
<td>6</td>
<td>33,3</td>
<td>50</td>
<td>2,920</td>
</tr>
<tr>
<td>7</td>
<td>30,0</td>
<td>119</td>
<td>4,791</td>
</tr>
<tr>
<td>8</td>
<td>34,6</td>
<td>86</td>
<td>4,032</td>
</tr>
<tr>
<td>9</td>
<td>33,0</td>
<td>39</td>
<td>2,294</td>
</tr>
</tbody>
</table>

The prior \(\pi(\mu)\) obtained using the transformation is shown in Fig 6b.

The new data \(y\) was obtained from a series of nine experiments which investigated the interaction of blast wave and circular embankment \(n = 9\). Elements of the sample \(y\) are hypothetical and given in Table 2. This sample was transformed into the sample of f.f. values, \(p_j\), by applying the aleatory f.f. \(F_j(y)\) into a set of \(m\) quantiles \(p_k\). As the d.f. \(F_j(p)\) is not known in the present case, the values

**Fig 7.** Likelihood function estimate \(L(\hat{\mu}_{450} | \mu)\) (solid line), prior \(\pi(\mu)\) (dash and line) and estimate of the posterior, \(\hat{\pi}(\mu | \hat{\mu}_{450})\) (dotted line) obtained with the bandwidth \(w = 0.1\).
\( p_j \) were estimated by the empirical quantiles \( \hat{p}_{1,k(m+1)} \) computed for the samples \( p_j' \), each consisting of 100,000 simulated values \( p_j \) of the r.v. \( F(y | \Theta) \). The discretization of \( p_j \) was carried out using two sets of the quantiles \( \hat{p}_{j,k(m+1)} \) created with \( m = 50 \) and \( m = 100 \).

The simulated samples \( p_j' \) were combined into the sample \( p^* \) consisting of 900,000 elements. Descriptive measures of \( p^* \) and \( p_j' \) are given in Tables 3 and 4.

The samples \( p_j' \) can be used to control the results of the discretization expressed by the samples of quantiles, \( \hat{p} \) and \( p_j' \). For instance, descriptive measures of the latter samples computed for the case \( m = 100 \) are given in Tables 5 and 6. Descriptive measures of \( p_j \) differ from the ones of \( p_j' \) to a relatively large extent (compare Tables 4 and 5). The transformation produced the samples \( p_j' \) which are closer to \( p_j \) in terms of their mean values, standard deviations, and skewnesses (compare Tables 4 and 6). Larger differences in the descriptive measures were obtained only in the cases \( j = 6 \) and \( j = 9 \), namely, in cases of a relatively large skewness of the samples \( p_6 \) and \( p_9 \). One can conclude that in case of highly skewed samples \( p_j' \), the transformation should be replaced by a more sophisticated one which will yield better adjustment of the samples \( p_j \) to the simulated samples \( p_j' \).

For the case \( m = 100 \), clustering the nine samples \( p_j' \) resulted in a sample \( p' \) containing 900 elements and

**Table 5.** Descriptive measures of the samples \( p_j \) obtained using the transformation (8) with \( m = 100 \)

<table>
<thead>
<tr>
<th>( j )</th>
<th>( y ) (kPa)</th>
<th>Mean</th>
<th>St.dev.</th>
<th>Skew.</th>
<th>Kurt.</th>
<th>Minimum</th>
<th>Maximum</th>
<th>10th prec.</th>
<th>90th prec</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.767</td>
<td>3,415 \times 10^{-4}</td>
<td>9.041 \times 10^{-4}</td>
<td>4.51</td>
<td>23.90</td>
<td>2,120 \times 10^{-9}</td>
<td>8,396 \times 10^{-7}</td>
<td>1026 \times 10^{-6}</td>
<td>8,622 \times 10^{-5}</td>
</tr>
<tr>
<td>2</td>
<td>4.276</td>
<td>1.153 \times 10^{-2}</td>
<td>2.505 \times 10^{-2}</td>
<td>3.73</td>
<td>16.43</td>
<td>7,397 \times 10^{-8}</td>
<td>1.624 \times 10^{-1}</td>
<td>1.568 \times 10^{-3}</td>
<td>3,203 \times 10^{-2}</td>
</tr>
<tr>
<td>3</td>
<td>4.160</td>
<td>8.066 \times 10^{-3}</td>
<td>1.948 \times 10^{-2}</td>
<td>3.89</td>
<td>17.79</td>
<td>3.195 \times 10^{-8}</td>
<td>1.285 \times 10^{-1}</td>
<td>8,488 \times 10^{-6}</td>
<td>2,366 \times 10^{-2}</td>
</tr>
<tr>
<td>4</td>
<td>3.944</td>
<td>5.273 \times 10^{-3}</td>
<td>1.309 \times 10^{-2}</td>
<td>4.28</td>
<td>21.57</td>
<td>7,107 \times 10^{-9}</td>
<td>9,057 \times 10^{-2}</td>
<td>2,711 \times 10^{-6}</td>
<td>1,365 \times 10^{-2}</td>
</tr>
<tr>
<td>5</td>
<td>4.916</td>
<td>4.069 \times 10^{-2}</td>
<td>6.671 \times 10^{-2}</td>
<td>2.73</td>
<td>8.51</td>
<td>3.934 \times 10^{-6}</td>
<td>3,755 \times 10^{-1}</td>
<td>3,054 \times 10^{-4}</td>
<td>1,157 \times 10^{-1}</td>
</tr>
<tr>
<td>6</td>
<td>2.920</td>
<td>3.219 \times 10^{-4}</td>
<td>1.165 \times 10^{-3}</td>
<td>5.95</td>
<td>40.74</td>
<td>1.638 \times 10^{-12}</td>
<td>9,409 \times 10^{-3}</td>
<td>4,141 \times 10^{-9}</td>
<td>5,742 \times 10^{-4}</td>
</tr>
<tr>
<td>7</td>
<td>4.791</td>
<td>3.258 \times 10^{-2}</td>
<td>5.651 \times 10^{-2}</td>
<td>2.92</td>
<td>9.87</td>
<td>1.935 \times 10^{-5}</td>
<td>3,274 \times 10^{-1}</td>
<td>1,752 \times 10^{-4}</td>
<td>9,242 \times 10^{-2}</td>
</tr>
<tr>
<td>8</td>
<td>4.032</td>
<td>6.544 \times 10^{-3}</td>
<td>1.572 \times 10^{-3}</td>
<td>4.11</td>
<td>19.85</td>
<td>1.347 \times 10^{-8}</td>
<td>1.064 \times 10^{-1}</td>
<td>4,247 \times 10^{-6}</td>
<td>1,750 \times 10^{-2}</td>
</tr>
<tr>
<td>9</td>
<td>2.294</td>
<td>4.403 \times 10^{-3}</td>
<td>1.929 \times 10^{-4}</td>
<td>6.86</td>
<td>52.74</td>
<td>3,886 \times 10^{-15}</td>
<td>1.659 \times 10^{-3}</td>
<td>4,219 \times 10^{-11}</td>
<td>5,402 \times 10^{-3}</td>
</tr>
</tbody>
</table>

**Table 6.** Descriptive measures of the samples \( p_j \) obtained by transforming the samples \( p_j \) by means of Eq. (16) (the latter samples result from the discretization of continuous distributions of r.v.s \( p_j \) at \( m = 100 \))

<table>
<thead>
<tr>
<th>( j )</th>
<th>( y ) (kPa)</th>
<th>( \lambda )</th>
<th>Mean</th>
<th>St.dev.</th>
<th>Skew.</th>
<th>Kurt.</th>
<th>Minimum</th>
<th>Maximum</th>
<th>10th prec.</th>
<th>90th prec</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.767</td>
<td>0.51</td>
<td>4.153 \times 10^{-3}</td>
<td>1.250 \times 10^{-2}</td>
<td>5.33</td>
<td>33.4</td>
<td>2,210 \times 10^{-9}</td>
<td>9,658 \times 10^{-2}</td>
<td>1026 \times 10^{-6}</td>
<td>9,215 \times 10^{-5}</td>
</tr>
<tr>
<td>2</td>
<td>4.276</td>
<td>0.28</td>
<td>1.283 \times 10^{-2}</td>
<td>3.008 \times 10^{-2}</td>
<td>4.19</td>
<td>21.0</td>
<td>7,397 \times 10^{-8}</td>
<td>2,079 \times 10^{-1}</td>
<td>1,568 \times 10^{-3}</td>
<td>3,380 \times 10^{-2}</td>
</tr>
<tr>
<td>3</td>
<td>4.160</td>
<td>0.35</td>
<td>9.830 \times 10^{-3}</td>
<td>2.442 \times 10^{-2}</td>
<td>4.45</td>
<td>23.6</td>
<td>3.195 \times 10^{-8}</td>
<td>1,735 \times 10^{-1}</td>
<td>8,488 \times 10^{-6}</td>
<td>2,519 \times 10^{-2}</td>
</tr>
<tr>
<td>4</td>
<td>3.944</td>
<td>0.43</td>
<td>6.211 \times 10^{-3}</td>
<td>1.725 \times 10^{-2}</td>
<td>4.99</td>
<td>29.5</td>
<td>7,107 \times 10^{-9}</td>
<td>1,295 \times 10^{-1}</td>
<td>2,712 \times 10^{-6}</td>
<td>1,453 \times 10^{-2}</td>
</tr>
<tr>
<td>5</td>
<td>4.916</td>
<td>0.16</td>
<td>4.327 \times 10^{-3}</td>
<td>7.416 \times 10^{-2}</td>
<td>2.95</td>
<td>10.2</td>
<td>3.934 \times 10^{-6}</td>
<td>4,356 \times 10^{-1}</td>
<td>3,054 \times 10^{-4}</td>
<td>1,214 \times 10^{-1}</td>
</tr>
<tr>
<td>6</td>
<td>2.920</td>
<td>0.95</td>
<td>4.680 \times 10^{-4}</td>
<td>2.068 \times 10^{-2}</td>
<td>7.26</td>
<td>58.9</td>
<td>1.638 \times 10^{-12}</td>
<td>1,835 \times 10^{-2}</td>
<td>4,141 \times 10^{-9}</td>
<td>6,075 \times 10^{-4}</td>
</tr>
<tr>
<td>7</td>
<td>4.791</td>
<td>0.18</td>
<td>3.490 \times 10^{-2}</td>
<td>6.363 \times 10^{-2}</td>
<td>3.18</td>
<td>12.0</td>
<td>1.935 \times 10^{-6}</td>
<td>3,864 \times 10^{-1}</td>
<td>1,752 \times 10^{-4}</td>
<td>9,712 \times 10^{-2}</td>
</tr>
<tr>
<td>8</td>
<td>4.032</td>
<td>0.39</td>
<td>7.598 \times 10^{-3}</td>
<td>2.021 \times 10^{-2}</td>
<td>4.75</td>
<td>26.7</td>
<td>1.347 \times 10^{-8}</td>
<td>1,479 \times 10^{-1}</td>
<td>4,247 \times 10^{-6}</td>
<td>1,863 \times 10^{-2}</td>
</tr>
<tr>
<td>9</td>
<td>2.294</td>
<td>1.60</td>
<td>8.142 \times 10^{-5}</td>
<td>4.595 \times 10^{-4}</td>
<td>8.35</td>
<td>75.0</td>
<td>3,886 \times 10^{-13}</td>
<td>4,313 \times 10^{-3}</td>
<td>4,219 \times 10^{-11}</td>
<td>5,683 \times 10^{-8}</td>
</tr>
</tbody>
</table>
having descriptive measures presented in Table 3.

The samples $p$ containing 450 and 900 elements were used to calculate the respective likelihood function estimates $L_B (\hat{\mu}_{450} \mid \mu)$ and $L_B (\hat{\mu}_{900} \mid \mu)$. The normalizing constants $C (\hat{\mu}_{450})$ and $C (\hat{\mu}_{900})$ found by a numerical integration are equal to 3,08868 and 3,089694, respectively. Fig 7 shows the graphs of the functions $\pi (\mu)$, $L_B (\hat{\mu}_{450} \mid \mu)$, and $\hat{\pi} (\mu \mid \hat{\mu}_{450})$.

The number of bootstrap replications, $B$, necessary to generate the sample $\{ \hat{\mu}_{450} \}$, was computed at the bandwidth $w = 0.1$. This value was chosen using the rule $w \propto B^{-1/3}$ proposed by Davison and Henley (1998: 277).

The results in the posterior distribution estimates, $\hat{\pi} (\mu \mid \hat{\mu}_{450})$ and $\hat{\pi} (\mu \mid \hat{\mu}_{900})$, were obtained by applying the Gaussian kernel function $k_\pi (.)$ (e.g. Davidson and Hinkley 1998: 168). The approximations of the posterior, $\hat{\pi} (\mu \mid \hat{\mu}_{450})$ and $\hat{\pi} (\mu \mid \hat{\mu}_{900})$, were computed at the bandwidth $w = 0.1$. This value was chosen using the rule $w \propto B^{-1/3}$ proposed by Davison and Henley (1998: 277).

The differences between the likelihood function estimates $L_B (\hat{\mu}_{450} \mid \mu)$ and $L_B (\hat{\mu}_{900} \mid \mu)$ is slight (Fig 8). This results in a slight difference between the posteriors, $\hat{\pi} (\mu \mid \hat{\mu}_{450})$ and $\hat{\pi} (\mu \mid \hat{\mu}_{900})$ (Fig 9). The random fluctuation of differences shown in Figs 8 and 9 is due to the application of the stochastic simulation to the computation of bootstrap samples. The means values of the samples $p'$ consisting of 450 and 900 elements are approximately equal, namely, $\hat{\mu}_{450} = 0.013234$ and $\hat{\mu}_{900} = 0.013261$ (Table 3). The mean values of the bootstrap samples $\hat{\mu}_{450}$ and $\hat{\mu}_{900}$ seem to be relatively close, no matter what is the size of $p'$.

The approximation of the posterior, $\hat{\pi} (\mu \mid \hat{\mu}_{450})$, expresses the updated epistemic uncertainty in the failure probability $P (F \mid d')$. Fig 7 indicates that $\hat{\pi} (\mu \mid \hat{\mu}_{450})$ is more accurate that the prior $\pi (\mu)$. The degree of “accuracy” can be expressed by the ranges of non-conservative and conservative percentiles given in Table 7. The new nine experimental records of the blast wave overpressure represented by the sample $y$ decreased the uncertainty expressed by the prior $\pi (\mu)$. One can anticipate that the decision-maker will understand the conservative percentiles shown in Table 7 better than the entire density.

Conclusions

An estimation of a probability of structural failure of steel tank due to an accidental action has been considered. The probability was estimated using two sources of evidence. The first was a small-size statistical sample consisting of experimental measurements of the accidental action characteristics. The second source was a fragility function used to express aleatory and epistemic uncertainties in the potential failure given fixed values of the action characteristics. The failure probability of steel tank was estimated by applying Bayesian reasoning. Epistemic uncertainty in the failure probability was expressed in the form of Bayesian prior and posterior distributions. The prior distribution was obtained by propagating epistemic uncertainty related to fragility function values and, if necessary, in values of demand variables. The posterior distribution was estimated by carrying out Bayesian updating with uncertain (imprecise) data. This data was expressed as a set of continuous epistemic probability distributions of fragility function values. The distributions were related to elements of the small-size sample.

The Bayesian updating with the set of continuous distributions of epistemic uncertainty is possible by discretizing these distributions. The discretization yields a new sample which can enter into Bayes theorem through a likelihood function. This sample consists of fragility function values, each of which has equal epistemic weight. Such a discretization replaces the continuous random variable by a set of non-uniformly discrete values. These values can be obtained by dividing the range of the inverse distribution function of each epistemic distribution into equal intervals. In case where the continuous epistemic distributions are highly skewed, an additional transformation of the discrete distribution can improve the discretization. The proposed scheme of discretization is an alternative to the widely-know posterior averaging approach. This approach is not directly applicable to the case considered in the paper.

The Bayesian estimation of the failure probability was illustrated by an example which considers a potential explosive damage to a steel tank structure. The probability of this damage was estimated in this example. A potential field of application of the proposed approach is the risk studies of hazardous industrial and transportation facilities like the studies presented in the articles Fabbrocino et al. (2005), Na and Shinozuka (2008).

References


