MATHEMATICAL MODEL OF PNEUMATIC SYSTEMS FOR SHOCK ABSORBING, TAKING INTO ACCOUNT CHARACTERISTICS OF ACTUAL GAS

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Abstract. The present work employs an approximation formula, allowing, with accuracy sufficient for technical calculations, computing value of coefficient of compressibility of actual gas at given temperature and pressure. The present mathematical model can also be used as a constituent of a more complete mathematical model of spatial vibrations of railway vehicles, where pneumatic shock absorbers and vibration dampers are used.

Keywords: actual gas, ideal gas, absorption, pneumatic systems, density, pressure, coefficient of compressibility, universal gas constant, computing algorithms, function.

1. Introduction

When calculating and modelling pneumatic actuating and controlling elements it is usual to employ ideal gas equation. The present work employs an approximation formula, allowing, with accuracy sufficient for technical calculations, computing value of coefficient of compressibility of actual gas at given temperature and pressure. It examines isentropic and adiabatic gas flow from space 1 with known parameters of actual gas state to space 2, which also has known parameters of gas state. The example shows essential difference between function of actual gas flow and function of ideal gas flow at different initial temperatures and pressures of gas.

2. Mathematical model

As it is known, for long time, in passenger car bogies combined systems of spring suspension were used, namely systems using springs (metal ones, as a rule) and hydraulic and friction shock absorbers. It would be interesting to use more modern systems of absorption, such as pneumatic suspension. Operational principle of these systems is based on use of air (gas or mixture of gases) as filling agent of elastic cylinders, which perform the function of absorbers, and their elastic characteristics are regulated by respective devices depending on parameters of car movement. Meanwhile one of important processes requiring examination is the process of gas (air) flow between two communicating chambers. Mathematical description of the process will allow more accurate evaluation of characteristics and parameters of entire pneumatic system as a whole in design stage.

Let us assume that the gas flows from space 1, characterised by temperature $T_1$, density $\rho_1$ and pressure $p_1$, to space 2, characterised by temperature $T_2$, density $\rho_2$ and density $p_2$. State of actual gas in each chamber is described by the following equations:

$$p_i = \frac{\rho_i R T_i}{\mu}, \quad i = 1, 2, \quad (1)$$

where $\mu$ is coefficient of compressibility of actual gas, and $R$ is the universal gas constant.

Literature [3–4] presents generalised curves of dependence of gas compressibility coefficient on pressure values at different temperatures. Application of these curves on a series of values of reduced pressure $c_{pr}$ and reduced temperature $c_{T}$ allowed obtaining numerical values of experimentally defined gas compressibility coefficient $z(p_r, T_r)$, as shown in Table 1. The pressure and temperature values are equal to relationships of pressure and temperature values to their critical values $T_c$ and $p_c$ – critical temperature, above which the gas cannot be transformed to liquid under any pressure, and $p_c$ – critical pressure; thus in case of nitrogen $T_c = 126 \, ^°K$, $p_c = 3, 39 \, MPa$ [8]. Values in brackets in Table 1 show pressure and temperature values for nitrogen, corresponding the selected values shown.

Experimental diagrams can be approximated.
with sufficient accuracy for practical technical calculations using the following expression:

\[
z(p_i, T_i) = 1 - 1.16e^{-0.7T_i} + \left( 0.012 + 0.16e^{-0.7T_i} \right)p_i + \left[ 0.27 + 0.057(p_i - 4)^2 \right] e^{-0.7T_i} (5.0 - p_i), \tag{2}
\]

where \( \sigma_d(5.0 - p_i) \) is Heaviside’s operator, equal to one when \( p_i \leq 5.0 \) and to zero when \( p_i > 5.0 \).

Table 1 presents values of \( z_{appr}(p_i, T_i) \) and errors for definition, with the help of approximating formula (2), of the coefficient of gas compressibility, showing that within limits of \( 0 \leq p_i \leq 40 \) and \( 1.8 \leq T_i \leq 6.0 \) (in case of nitrogen – from \(-50°C \) to \(+483°C \)) the accuracy of calculation of gas compressibility coefficient with the help of approximating expression (2) is sufficient when performing technical calculations.

Table 1. Value of the Gas Compressibility Coefficient

<table>
<thead>
<tr>
<th>( T_i ) (°C)</th>
<th>( P_i ) (p, MPa)</th>
<th>( Z(p_i, T_i) )</th>
<th>( z_{appr}(p_i, T_i) )</th>
<th>( % )</th>
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<td>2,96</td>
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At the same time it must not be forgotten that error of definition of gas compressibility coefficient according to experimentally obtained diagrams is commensurate with the approximation error.

It should also be noted that the approximating expression (2) is built formally mathematically and is not based on any physical assumptions.

In order to define flow rate over channel interconnecting the spaces 1 and 2, we will use the Bernoulli’s equation, taken for constant voluminous potential forces, in terms of

\[
\frac{v_i^2}{2} + P(p_i(t)) = \text{Const} \tag{3}
\]

In this Bernoulli’s equation, \( v_i \) is rate of gas flow from chamber \( i \), and \( P(p_i) \) is a pressure function (barotropic movement)

\[
P(p_i) = \int \frac{dp_i}{b_i(p)}, \tag{4}
\]

\( p(p) \) is dependence of gas density in respect of pressure. The dependency is defined by the process in question; \( p_i \) is a certain initial pressure in space No \( i \), and \( p_i(t) \) is a time-variant gas pressure at its flow-over from space No \( i \).

To formulate the said dependencies, we will use a somehow simplified equation of Van der Waal’s, presented as

\[
p_i \left[ 1 - b_i \frac{p_i}{\mu} \right] = \frac{p_i}{\mu} RT_i, \quad i = 1, 2, \tag{5}
\]

where \( \mu \) is a weight of one mole of gas and value of the \( b_i \) coefficient is defined (as it follows from expressions (5) and (1)) at every moment of time with the help of the following expression

\[
b_i = \frac{z_{i} - 1}{p_i} RT_i, \quad i = 1, 2, \tag{6}
\]

where \( z_{i} \) is a known experimentally defined value of the gas compressibility coefficient for its state defined by temperature \( T_i \) in a given moment and average value of pressure \( p_i \) in interval \([p_i, p_i(t)]\).

Further let’s examine the short-term process of gas flow from chamber 1 to chamber 2 as isoentropic and adiabatic. It can be shown that in adiabatic process the following equation will be true:

\[
p_i(t) \left( \frac{\mu}{p_i(t)} - b_i \right)^k = p_i(t) \left( \frac{\mu}{p_i} - b_i \right)^k, \tag{7}
\]

where for diatomic gases \( k = 1, 4, \) and \( p_i \) and \( p_i(t) \) are values of gas pressure and density as fixed for a certain initial moment of time, in a chamber.
corresponding to the index number. From this we obtain:

$$
\rho_i(t) = \frac{\mu \rho_i}{\left( \frac{p_i(t)}{\rho_i} \right)^{\frac{1}{k}}} - b \rho_i \left( 1 - \left( \frac{p_i(t)}{\rho_i} \right)^{\frac{1}{k}} \right),
$$

(8)

By inserting the latter expression into (4) and then, after integration, into (3), after a series of modifications we will obtain the following

$$
\frac{u_2^2}{2} = k \cdot \frac{p_i}{k - 1} \rho_i \left( 1 - \frac{b \rho_i}{\mu} \left( 1 - \frac{k - 1}{\sigma_k} \right) \right) - b \rho_i \left( 1 - \sigma \right) = \text{Const} = \frac{u_1^2}{2},
$$

(9)

where \( \sigma = \frac{p_2}{p_1} \). Initial speed of gas in the chamber, from which it flows, can be considered as equal to zero, namely \( u_1 = 0 \). This way,

$$
u_2 = \left[ \frac{2k \cdot \frac{p_i}{k - 1} \rho_i \left( 1 - \frac{b \rho_i}{\mu} \left( 1 - \frac{k - 1}{\sigma_k} \right) \right)}{b \rho_i \left( 1 - \sigma \right)} \right]^{\frac{1}{2}}.
$$

(10)

In case of ideal gas, when \( b_i = 0 \), the expression (10) is reduced to the familiar equality of Saint Venant and Vantzel.

Having found the gas flow rate, let’s calculate the flow \( G(t) \) according to formula

$$
G(t) = \xi^2 \int_\tau^\sigma p_2 \cdot u_2, \quad \xi
$$

(11)

where \( \xi \) is resistance coefficient of gas flow channel, determined experimentally and dependent on channel form and resistance to gas movement over the channel (acceleration channels and de Laval type nozzles are not considered here), and \( f \) is channel section area, over which the gas flows from one chamber to another.

Gas density in chambers can be calculated with the help of expression (7). More accurate determination of gas density in each chamber can be defined with the help of expression

$$
\rho_i = \frac{\mu \nu_i}{V_i}, \quad i = 1, 2,
$$

(12)

where \( V_i \) is a cavity space, and \( \nu_i \) is quantity of gas moles in the chamber. For determination of gas quantity in final volume chambers and gas temperatures therein, additional differential equations must be solved.

Having expressed the value of \( \rho_2 \) by means of \( \rho_1 \) value at given size of \( \sigma \) and inserted it, together with expression (9), into expression (10), we will obtain, after a series of modifications, the formula for actual gas flow at its overflow from chamber 1 to chamber 2 under lesser pressure:

$$
G = f p_1 \left[ \frac{\mu \left( \frac{RT_i + b_i \rho_i \sigma^k}{\xi} \right)}{\frac{2k}{k - 1} \left( 1 - \sigma \right) \left( \frac{2}{\sigma_k^k} - \frac{k + 1}{\sigma_k^k} \right) + b_i \rho_i \left( \frac{2}{\sigma_k^k} - \frac{k + 2}{\sigma_k^k} \right)} \right]^{\frac{1}{2}},
$$

(13)

where \( b_i = \frac{\sigma - 1}{p} RT_i \), and \( \xi \) is the value of gas compressibility coefficient at pressure \( p = (p_1 + p_2) / 2 \) and temperature \( T_i \).

Analysis of change of flow \( G \) in dependence of reduction of value \( \sigma \), as carried out in references [1–2] for ideal gas, shows that at a certain critical value \( \sigma_{cr} = 0, 528 \), maximum value of \( G \) is achieved. Physically this is explained by the fact that at such value of correlations of gas pressures at inlet and outlet of flow, flow rate equal to sound speed is found. At such a speed, all the pressure reductions at \( \sigma < \sigma_{cr} \) cannot distribute against flow and will drift to flow direction without changing its rate. That is, when \( \sigma < \sigma_{cr} = 0,528 \) the flow rate will remain constant and equal to maximum value. Analogous phenomena will take place also in the present case of actual gas flow. At the same time one must not forget that the sound speed significantly depends on pressure and temperature of gas. Therefore, when building the model with the use of above formulae, algorithm for calculation of flow must contain numerical analysis of values of flow \( G \) at the maximum under conditions at the moment of time \( t \).
Figure 1 qualitatively shows the dependence diagram $G(\sigma)$. Dotted part of the curve, corresponding to values $G$ when $\sigma < \sigma_{кр}$, is replaced in calculations with horizontal section $G = G_{max}$.

Let’s examine one of the algorithms for calculating function $G(t)$. It is suggested that for the given moment we know all the values belonging to the formulae (10)–(12) or (13), that is, for the given moment $t$ we know pressure values $p_i$ and $\overline{p} = 0,5(p_1 + p_2)$, corresponding value $\overline{z}$, value $\sigma$ and its corresponding value $G$, calculated with the help of expressions (10)–(12) or (13). Let’s designate this value as $G_0$. Then we will increase the value $\sigma$ by small quantity $\sigma \Delta$, namely we will calculate a new value $\sigma \Delta + \sigma$ and its corresponding value $G_1$ with the help of expressions (10)–(12) or (13), to which we will assign symbol $G_1$. If it turns out that $G_1 < G_0$, we will assume that $G(t) = G_0$. But if it turns out that $G_1 > G_0$, then again we will increase the previous value $\sigma$ by $\sigma \Delta$ and calculate $G$ with the help of expressions (10)–(12) or (13) and equate it to $G_1$. Then we produce a comparison, as in previous cycle, with corresponding conclusions. Recurrent process of calculations is performed at every moment of time, until the calculations result in value $G(t)$, corresponding to solid line in Fig. 1. Figure 2 shows a flow graph for calculation of value $G(t) = G(\sigma(t))$ with the use of the described algorithm.

Pressure can be calculated most accurately using the expression (1), if gas compressibility coefficients are known, or expression (2).

Further let’s examine impact of actual gas properties on its flow-over from one chamber to another in comparison with flow-over of ideal gas. Expression (13) can be modified into

$$G = G_0 g(\sigma, p_{tr}, T_{tr}),$$

where

$$G_0 = fp_1 \left( \frac{\mu}{\varepsilon RT} \right)^{\frac{1}{2}},$$

and $g(\sigma, p_{tr}, T_{tr})$ is a dimensionless flow function, determined from expression

$$g = \left( 1 + \beta_1 \sigma^2 \left( \frac{2k}{(1 + \beta_1)^2} \left( \frac{2\kappa + 1}{2\kappa - \sigma} \right) \right)^\frac{1}{2} \right) + \beta_1 \left( \frac{2\kappa}{(1 + \beta_1)^2} \left( \frac{2\kappa + 1}{2\kappa - \sigma} \right) \right)^\frac{1}{2},$$

where

$$\beta_1 = \frac{2}{1 + \sigma} \left[ \left( 0,006 + 0,08 \exp(-0,7T_{tr}) \right) \left( 1 + \sigma \right) p_{tr} - 1,16 \exp(-0,7T_{tr}) + 0,27 + 0,057 \left( 0,5(1 + \sigma) p_{tr} - 4 \right)^2 \exp(-0,7T_{tr}) \right] \times g(5,0 - 0,5(1 + \sigma) p_{tr})$$

Figure 3 presents, as an example, the dependencies of flow functions $g(\sigma)$ in case of examination of ideal gas (the diagram does not depend on values of pressure and temperature ion chamber 1) and actual gas at different pressures and temperature of 20 °C.

Figure 1 shows that in case of actual gas the pressure in chamber 1 makes impact both on critical value of relation $\sigma$ and on value of flow function when values of $\sigma$ are lower than the critical value.

In order to examine impacts of pressure and temperature on flow functions $g(\sigma)$ of actual gas, definition and calculation was made for value
Dependence of these values on given pressure $p_{1r}$ in chamber 1 for various temperatures is shown in Fig. 4 and Fig. 5 accordingly. After analysis of Figures 4 and 5 it must be noted that in case of actual gas the critical value $\sigma_{cr}$ of relation of gas pressure in chambers 1 and 2 is different from corresponding value $\sigma_{cr}$ for ideal gas by not more than 15%. At the same time, the value flow function in case of actual gas when $\sigma < \sigma_{cr}$ can differ from significantly (almost twice), which fact must be taken into account in design of pneumatic actuating and controlling elements, especially in systems of pneumatic springs and pneumatic shock absorbers.

All the calculations were made with the help of MatLab 7 software.

Also for adiabatic flow-over of gas, the author has participated in development of analogous mathematical model, which is described in [6], and in case of non-adiabatic flow-over of gas the mathematical modelling can be performed in accordance with the model described in [7].

3. Conclusions

Thus we have suggested a description of mathematical model of actual gas flow-over in various pneumatic systems, which are used in railway transport. The present mathematical model can also be used as a constituent of a more complete mathematical model of spatial vibrations of railway vehicles, where pneumatic shock absorbers and vibration dampers are used.
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