ISSN 1392-8619 print/ISSN 1822-3613 online
ŪKIO TECHNOLOGINIS IR EKONOMINIS VYSTYMAS
TECHNOLOGICAL AND ECONOMIC DEVELOPMENT OF ECONOMY
http://www.tede.vgtu.lt
2007, Vol XIII, No 2, 93–99

PRISITAIKANČIŲJŲ SANTVARŲ OPTIMIZAVIMO UŽDAVINIŲ MATEMATINIAI MODELIAI JUDAMOSIOS APKROVOS ATVEJU

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Įteikta 2007-02-22; priimta 2007-06-07

Santrauka. Prisitaikomumo teorija, nagrinėjanti tampriai plastiškas konstrukcijas, veikiama kintamosios kartotinės apkrovos, leidžia judamąją apkrovą traktuoti kaip atskirą kartotinai kintančių jėgų atvejį. Apkrovai leidžiama „judėti" bet kuria konstrukcijos dalimi: nuo tilto vidurio, grįžti atgal, vėl į priekį – taip universaliai įvertinama apkrovimo istorija, kuri yra lemiamas faktorius, nagrinėjant plastines deformacijas patiriančios konstrukcijos įtempių ir deformacijų būvį. Straipsnyje atskleista galimybė taikyti prisitaikomumo teorijos metodus, sudarant teorinius santvarų optimizavimo uždavinių matematinius modelius ir juos sprendžiant. Nagrinėjama idealiai tampriai plastinė žinomos geometrijos santvara, veikiama judamosios apkrovos. Sudaryti minimalaus tūrio santvaros ar ją veikiančios apkrovos maksimizavimo uždavinių matematiniai modeliai. Modeliuose įvertinamos ne tik konstrukcijos stiprumo (prisitaikomumo) ir standumo sąlygos, bet ir stabilumo netekimo galimybė esant plastinei santvaros darbo stadijai. Pasiūlyti nauji sprendimo algoritmai, pateikti skaitiniai strypų lankstinės santvaros, veikiamos judamosios apkrovos, optimizavimo uždavinių pavyzdžiai. Tyrimai atlikti, darant mažų poslinkių prielaidą.

Reikšminiai žodžiai: prisitaikomumas, optimalus projektavimas, matematinis programavimas, idealiai tampriai plastinė santvara, judamoji apkrova.

MATHEMATICAL MODELS FOR OPTIMAL SHAKEDOWN TRUSSES DESIGN PROBLEMS IN CASE OF MOVING LOAD

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Received 22 February 2007; accepted 7 June 2007

Abstract. The shakedown theory, which analyses elastic-plastic constructions, subjected by variable repeated load, enables treating moving load as a separate type of variable repeated load. The load is allowed to „move" at any part of the construction: from the middle of the bridge, to turn back, again move ahead – in this manner loading history is universally evaluated and it is a crucial factor, considering stress-deformation state of structures under plastic deformations. This paper reveals a possibility to apply methods of shakedown theory for creation and solution of theoretical optimization mathematical models of trusses. The perfectly elastic-plastic loaded by moving load truss is considering. The mathematical models of the minimal volume truss or it acting load maximization problems are created. There are evaluating not only strength (shakedown) and rigidity restrictions, but also stability restriction in case of plastic state of truss in models. There is proposed new solution algorithms and introduced numerical examples of truss optimization in case of moving load. The results are valid for the small displacement assumptions.

Keywords: shakedown, optimal design, mathematical programming, perfectly elastic-plastic truss, moving load.

1. Santvaros optimizavimo uždavinių matematiniai modeliai

1.1. Problemos formulavimas

Pagrindinis statybinių konstrukcijų skaičiavimo tikslas – apskaičiuoti dėl išorinio poveikio atsirandančias įrąžas bei poslinkius ir, juos žinant, suprojektuoti pakankamai stiprius, standžius ir stabilius statinius. Konstrukcijų skaičiavimo uždavinys gali būti sprendžiamas, kai žinomos statinį veikiančios apkrovos, jų geometrija ir medžiagos. Statybinės mechanikos uždavinys, kuriame visi šie trys parametrai žinomi, paprastai vadinamas *analizės uždaviniu* [1]. Taip jis vadinamas todėl, kad sprendžiant tokį uždavinį nustatomas vien konstrukcijos įtempių ir deformacijų būvis, t. y. konstrukcija analizuojama mechaniniu požiūriu: lyginant su eksploataciniais reikalavimais, įsitikinama, ar konstrukcija yra pakankamai stipri, standi ar stabili [2].

Jeigu kurie nors iš išvardytų parametrų iš anksto nenustatyti, uždavinys tampa neapibrėžtas, jam išspręsti reikia papildomų sąlygų. Tenka jau ne tik analizuoti konstrukciją, bet, nustačius vienokias ar kitokias konstrukcijos parametrų ribas (nelygybėmis suformulavus stiprumo, standumo ar stabilumo sąlygas), siekti pasirinkto tikslo (ieškoti tinkamos apkrovos, tinkamos geometrijos ar tinkamos medžiagos). Taigi *optimizavimo uždavinio* tikslas – nustatyti optimalius tam tikro kriterijaus požiūriu nagrinėjamos konstrukcijos parametrus ar statinį veikiančios apkrovos pasiskirstymus [3–5].

Statybinės mechanikos optimizavimo uždaviniai yra įžanginis konstrukcijų optimalaus projektavimo etapas, pagrįstas deformuojamo kūno mechanikos lygtimis ir matematinio programavimo teorija, jos metodais bei jų mechanine interpretacija. Norint skaičiavimą pagrįsti realioms konstrukcijos darbo sąlygoms, būtina analizės ir optimizavimo uždavinių matematiniuose modeliuose kuo tiksliau įvertinti konstrukcijos medžiagos savybes ir išorinius poveikius. Iš dalies tai pasiekiama, apimant plastines medžiagos savybes, kuriomis pasižymi nemaža statybinių konstrukcijų, ypač metalinių [6–8].

Konstrukcijų skaičiavimas ir projektavimas, įvertinant plastines deformacijas, leidžia efektyviau išnaudoti jų laikomąją galią ir sudaryti ekonomiškesnius projektus [9]. Kita vertus, realūs konstrukcijos poveikiai dažniausiai yra cikliški. Kintamai kartotinė apkrova (KKA) – tai sistema jėgų, kurių kiekviena ar jų grupės gali kisti nepriklausomai viena nuo kitos. Tolesniuose svarstymuose KKA laikoma kvazistatine. Labai dažnai KKA nusakoma ne konkrečia apkrovimo istorija (kitimo laike dėsniu $\boldsymbol{F}(t)$), o tik viršutinėmis $\boldsymbol{F}_{sup}$ ir apatinėmis $\boldsymbol{F}_{inf}$ savo kitimo ribomis [10]: $\boldsymbol{F}_{inf} \le \boldsymbol{F}(t) \le \boldsymbol{F}_{sup}$.

Judamoji apkrova gali būti interpretuota kaip atskiras KKA atvejis [11]. Todėl santvaroms, veikiamoms judamosios apkrovos, optimizuoti galima taikyti tampriai plastinių prisitaikančiųjų konstrukcijų teorijos principus. Minimalaus santvaros tūrio projektas, gautas neatsižvelgus į standumo ir stabilumo apribojimus, dažniausiai neatitinka statybinėms konstrukcijoms keliamų eksploatacinių reikalavimų. Darbe santvaros strypų stabilumo apribojimai siejami su „Eurokodo 3“ rekomendacijomis, kai leistinosios ribinės gniuždomų strypų įrąžos gaunamos sumažinus tokių strypų takumo įtempius [6, 12, 13].

Straipsnyje sudarytų minimalaus tūrio santvarų ar optimalios apkrovos radimo uždavinio matematinių modelių tiesioginei realizacijai sukurti nauji algoritmai [14], leidžiantys metalinių santvarų skerspjūviams optimizuoti taikyti šiuolaikines kompiuterines technologijas. Tai iš dalies leidžia sugretinti realaus santvarų projektavimo ir gamybos rezultatus su teorinių paieškų bandomaisiais rezultatais [15].

1.2. Minimalaus tūrio santvaros uždavinys

Nagrinėjamas prisitaikiusios idealiai tampriai plastinės santvaros būvis. Santvaros geometrija (strypų ilgiai L_j, $j=1,2,...,n$, $j\in J$), medžiagos takumo riba σ_{yj}, tamprumo modulis E_j, apkrova duoti. Kintamos kartotinės apkrovos vektoriaus $\boldsymbol{F}(t)=(F_1(t), F_2(t), F_m(t))^T$ komponentai yra laike *t* kintančios jėgos, kurių pridėjimo vieta žinoma. Kiekviena jėga F_i charakterizuojama nepriklausančiomis nuo laiko *t* viršutinėmis ir apatinėmis kitimo ribomis $F_{i,sup}$, $F_{i,inf}$, $i=1,2,...,m$ ($i\in I$).

Minimalaus svorio santvaros projektas randamas sprendžiant uždavinį [16]:

rasti

$$\min \sum_j L_j A_j\ , \tag{1a}$$

kai

$$\boldsymbol{f}_{\max} = \boldsymbol{N}_0 - [G]\,\boldsymbol{\Theta}_p - \boldsymbol{N}_{e,\max} \ge \boldsymbol{0}, \tag{1b}$$

$$\boldsymbol{f}_{\min} = \boldsymbol{N}_{0,cr} + [G]\boldsymbol{\Theta}_p + \boldsymbol{N}_{e,\min} \ge \boldsymbol{0}, \tag{1c}$$

$$\boldsymbol{N}_0 = (N_{0j})^T, \quad \boldsymbol{N}_{0,cr} = (N_{0j,cr})^T,$$

$$N_{0,j} = \sigma_{yj} A_j, \quad N_{0,j,cr} = \varphi_j \sigma_{yj} A_j, \tag{1d}$$

$$A_j \ge A_{j,\min}, \quad j\in J, \tag{1e}$$

$$\boldsymbol{\Theta}_p = \boldsymbol{\lambda}_{\max} - \boldsymbol{\lambda}_{cr}, \tag{1f}$$

$$\boldsymbol{\lambda}_{\max}^T \boldsymbol{f}_{\max} = 0, \quad \boldsymbol{\lambda}_{cr}^T \boldsymbol{f}_{\min} = 0, \tag{1g}$$

$$\boldsymbol{\lambda}_{\max} \ge \boldsymbol{0}, \quad \boldsymbol{\lambda}_{cr} \ge \boldsymbol{0}, \tag{1h}$$

$$\boldsymbol{u}_{r,\min} \le \min[H]\boldsymbol{\Theta}_p, \quad \max\,[H]\boldsymbol{\Theta}_p \le \boldsymbol{u}_{r,\max}. \tag{1i}$$

Apkrovos kitimo ribų vektoriai $\boldsymbol{F}_{\mathrm{inf}}$ ir $\boldsymbol{F}_{\mathrm{sup}}$ žinomos, todėl ekstreminių jėgų vektoriai $\mathbf{N}_{\mathrm{e,max}}$ ir $\mathbf{N}_{\mathrm{e,min}}$, esantys tiesinėse takumo sąlygose $\boldsymbol{f}_{\max}$ (1b) ir $\boldsymbol{f}_{\min}$ (1c), matematiniame modelyje (1a–1i) yra žinomi (jų skaičiavimas detaliau paaiškintas antrame skyriuje). Tikslo funkcija (1a) formuojama pasitelkus strypų ilgius L_j ir skerspjūvių plotus A_j $(j \in J)$. Į standumo sąlygas (1i) nesudėtinga įtraukti ir tampraus santvaros skaičiavimo poslinkius, naudojant poslinkių influentinę matricą $[\beta]= \left([A][D]^{-1}[A]^T\right)^{-1}$, apkrovų ribų vektorius $\boldsymbol{F}_{inf}$ ir $\boldsymbol{F}_{sup}$, čia $[A]$ yra statikos lygčių koeficientų matrica, o $[D]$ – santvaros pasidavumo matrica.

Gniuždomų santvaros strypų galimas stabilumo netekimas įvertinamas takumo sąlygose (1c) ėmus naudoti redukuotą ribinių ašinių jėgų vektorių $\boldsymbol{N}_{0,cr}$. Vektoriaus $\boldsymbol{N}_{0,cr}$ komponentai $N_{0,j,cr}$ visiems $j \in J$ skaičiuojami, vadovaujantis „Euronormų 3“ (EN3) rekomendacijomis [12]:

$$N_{0,j,cr} = \varphi_j N_{0,j}, \text{ čia } \varphi_j = \frac{1}{\Phi_j + \left[\Phi_j^2 - \bar{\lambda}_j^2\right]^{0,5}}, \text{ kai}$$

$$\Phi_j = 0{,}5\left(1 + a\left(\bar{\lambda}_j - 0{,}2\right) - \bar{\lambda}_j^2\right), \quad \bar{\lambda}_j = \frac{\lambda_j}{\lambda_{1j}}\sqrt{\beta_A} = \frac{\lambda_j}{\pi\left[E_j / \sigma_{y,j}\right]^{0,5}}\sqrt{\beta_A},$$

$j \in J$, čia E_j yra *j*-ojo strypo tamprumo modulis; $\lambda_j = L_j / i_j$ – strypo liaunis, i_j – santvaros *j*-ojo strypo inercijos spindulys. Vien tik gniuždomų strypų atveju koeficientas $\beta_A = 1$, koeficientas *a*, įvertinantis strypų netobulumą, priklauso nuo skerspjūvio formos bei medžiagos savybių. Strypinės sistemos galimas stabilumo netekimas neįvertinamas, kai $\boldsymbol{N}_{0,cr} = \boldsymbol{N}_0$.

Netiesinio matematinio programavimo uždavinyje (1a)–(1i) nežinomaisiais yra santvaros elementų skerspjūvių plotai A_j, $j \in J$ ir plastinių daugiklių vektoriai $\boldsymbol{\lambda}_{\max}$, $\boldsymbol{\lambda}_{cr}$, kurie formuoja plastinių deformacijų vektorių $\boldsymbol{\Theta}_p = \boldsymbol{\lambda}_{\max} - \boldsymbol{\lambda}_{cr}$. Takumo sąlygose (1b) ir (1c) esanti sandauga $[G]\boldsymbol{\Theta}_p$ išreiškia liekamąsias įrąžas, čia $[G]$ – liekamųjų įrąžų influentinė matrica. Konstrukciniuose apribojimuose (1e) $A_j \geq A_{j,\min}$ naudojamos minimaliosios skerspjūvių plotų reikšmės $A_{j,\min}$. Formulės (1g), (1h) išreiškia matematinio programavimo griežtumo sąlygas. Konstrukcijos standumo apribojimai (1i) realizuojami, ribojant mazgų poslinkius ($\boldsymbol{u}_{r,\min}$, $\boldsymbol{u}_{r,\max}$ – duotieji liekamųjų poslinkių $\boldsymbol{u}_r = [H]\boldsymbol{\Theta}_p$ komponentų kitimo apatinių ir viršutinių ribų vektoriai, kur $[H]$ – liekamųjų poslinkių influentinė matrica). Būtent standumo sąlygos (1i), reikalaujančios papildomai spręsti tiesinį programavimo uždavinį [17], rodo, kad pagrindinis netiesinis santvaros optimizavimo uždavinys nėra klasikinis matematinio programavimo uždavinys. Todėl jis turi būti sprendžiamas etapais, apie sprendimo algoritmą bus kalbama trečiame skyriuje.

Tikslo funkcijos (1a) minimali reikšmė randama, neatsižvelgiant į galimą strypų stabilumo netekimą, jeigu matematinio modelio (1a)–(1i) takumo sąlygose (1c) takumo įtempimų mažinimo koeficientas $\varphi_j = 1$ $j \in J$.

1.3. Santvaros apkrovos optimizavimo uždavinys

Apkrovos kitimo ribų $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$ nustatymo (patikrinamasis) uždavinys, formuluojamas taip: *ieškomos prisitaikomumo būvio apkrovos kitimo ribos* $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$, *atitinkančios nustatytą optimalumo kriterijų* $\max\left\{\boldsymbol{T}_{sup}^{\mathrm{T}}\boldsymbol{F}_{sup} - \boldsymbol{T}_{inf}^{\mathrm{T}}\boldsymbol{F}_{inf}\right\}$ *bei konstrukcijos stiprumo, standumo ir stabilumo reikalavimus*, čia $\boldsymbol{T}_{sup}$, $\boldsymbol{T}_{inf}$ – optimalumo kriterijaus svorio koeficientų vektoriai.

Santvaros apkrovos optimizavimo prisitaikomumo sąlygomis uždavinys užrašomas taip:
rasti

$$\max\left\{\boldsymbol{T}_{\mathrm{sup}}^{\mathrm{T}}\boldsymbol{F}_{\mathrm{sup}} - \boldsymbol{T}_{\mathrm{inf}}^{\mathrm{T}}\boldsymbol{F}_{\mathrm{inf}}\right\}, \tag{2a}$$

kai

$$\boldsymbol{f}_{\max} = \boldsymbol{N}_0 - [G]\boldsymbol{\Theta}_p - \boldsymbol{N}_{e,\max} \geq \boldsymbol{0}, \tag{2b}$$

$$\boldsymbol{f}_{\min} = \boldsymbol{N}_{0,cr} + [G]\boldsymbol{\Theta}_p + \boldsymbol{N}_{e,\min} \geq \boldsymbol{0}, \tag{2c}$$

$$\boldsymbol{N}_0 = \left(N_{0j}\right)^T, \quad \boldsymbol{N}_{0,cr} = \left(N_{0j,cr}\right)^T,$$

$$N_{0,j} = \sigma_{yj}A_j, \quad N_{0,j,cr} = \varphi_j\sigma_{yj}A_j, \tag{2d}$$

$$\boldsymbol{F}_{\mathrm{sup}} \geq \boldsymbol{0}, \quad -\boldsymbol{F}_{\mathrm{inf}} \geq \boldsymbol{0}, \tag{2e}$$

$$\boldsymbol{\Theta}_p = \boldsymbol{\lambda}_{\max} - \boldsymbol{\lambda}_{cr}, \tag{2f}$$

$$\lambda_{\max}^T \boldsymbol{f}_{\max} = 0, \quad \lambda_{cr}^T \boldsymbol{f}_{\min} = 0, \tag{2g}$$

$$\lambda_{\max} \geq \boldsymbol{0}, \quad \lambda_{cr} \geq \boldsymbol{0}, \tag{2h}$$

$$\boldsymbol{u}_{r,\min} \leq \min[H]\Theta_p, \quad \max[H]\Theta_p \leq \boldsymbol{u}_{r,\max}. \tag{2i}$$

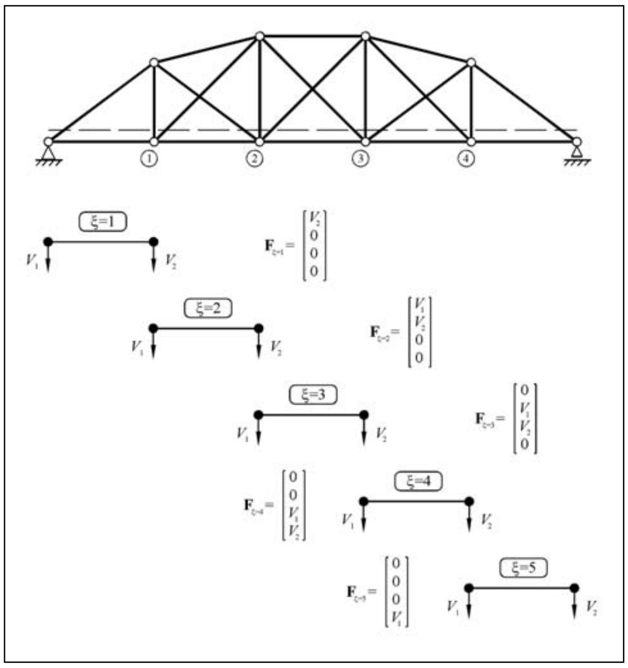
Ribinių ašinių jėgų vektoriai $\boldsymbol{N}_0$, $\boldsymbol{N}_{0,cr}$ ir liekamųjų poslinkių ribos $\boldsymbol{u}_{r,\min}$, $\boldsymbol{u}_{r,\max}$ yra žinomi dydžiai uždavinyje (2a)–(2i). Uždavinio (2a)–(2i) optimalus sprendinys yra vektoriai $\boldsymbol{F}_{\mathrm{sup}}^*$, $\boldsymbol{F}_{\mathrm{inf}}^*$ ir $\boldsymbol{\lambda}_{\max}^*$, $\boldsymbol{\lambda}_{cr}^*$.

2. Ekstreminių ašinių jėgų vektorių $\boldsymbol{N}_{e,\max}$ ir $\boldsymbol{N}_{e,\min}$ sudarymas

Vektoriams $\boldsymbol{N}_{e,\max}$ ir $\boldsymbol{N}_{e,\min}$ skaičiuoti reikalinga ašinių jėgų influentinė matrica $[\alpha]$. Šių vektorių sudarymas aptariamas 1 pav.

Dviejų jėgų sistema V_1 ir V_2 juda apatine santvaros juosta ir gali užimti keturias padėtis ties mazgais 1, 2, 3 ir 4. Bendruoju atveju padėčių gali būti $\xi = 1, 2, \ldots, p$ ($\xi \in P$) ir kiekviena padėtis charakterizuojama savo apkrovos vektoriumi $\boldsymbol{F}_\xi$ (čia pravartu prisiminti formulę $\boldsymbol{N}_{e,\xi} = [\alpha]\boldsymbol{F}_\xi$). Ateityje paprastumo dėlei naudojamas ne pilnutinis apkrovos vektorius **F** (tiksliau kalbant, vektorius $\boldsymbol{F}_\xi$), o jo pavektoris $\hat{\boldsymbol{F}}_\xi$, susietas tik su santvaros važiuojamąja dalimi. Pavyzdžiui, nagrinėjamai santvarai sudaromi penki apkrovos vektoriai (kiekvienai apatine santvaros juosta jėgų sistemos V_1 ir V_2 padėčiai):

$$\hat{\boldsymbol{F}}_1 = (V_2, \ 0, \ 0, \ 0)^T, \qquad \hat{\boldsymbol{F}}_2 = (V_1, \ V_2, \ 0, \ 0)^T,$$



1 pav. Judamoji apkrova užrašyta vektoriais $\boldsymbol{F}_\xi$

Fig 1. Moving load realized by vectors $\boldsymbol{F}_\xi$

$\hat{\boldsymbol{F}}_3 = (0,\ V_1,\ V_2,\ 0)^T$, $\hat{\boldsymbol{F}}_4 = (0,\ 0,\ V_1,\ V_2)^T$, $\hat{\boldsymbol{F}}_5 = (0,\ 0,\ 0,\ V_1)^T$. Dabar pagal formulę

$$\boldsymbol{N}_{e,\xi} = [\hat{\alpha}]\hat{\boldsymbol{F}}_\xi \tag{3}$$

skaičiuojamos pseudotamprios įrąžos kiekvienai judamosios jėgų sistemos padėčiai $\xi \in P$ (matrica $[\hat{\alpha}]$ yra influentinės matricos $[\alpha]$ pamatricė, abiejų matricų eilučių skaičius vienodas). Kiekvienas vektorių $\boldsymbol{N}_{e,\max} = (\mathbf{N}_{e1,\max},\ \mathbf{N}_{e2,\max},\ \ldots,\ \mathbf{N}_{en,\max})^T$ ir $\boldsymbol{N}_{e,\min} = (\mathbf{N}_{e1,\min},\ \mathbf{N}_{e2,\min},\ \ldots,\ \mathbf{N}_{en,\min})^T$ komponentas skaičiuojamas pagal formules:

$$\mathbf{N}_{ej,\max} = \max_{\xi} \mathbf{N}_{ej,\xi},\quad \mathbf{N}_{ej,\min} = \min_{\xi} \mathbf{N}_{ej,\xi}$$
$$\text{visiems } \xi \in P \text{ ir } j \in J. \tag{4}$$

Taigi, santvaros uždavinių (1a)–(1i), (2a)–(2i) takumo sąlygose (1b)–(1c), (2b)–(2c) įrašytos visos tampraus skaičiavimo įrąžos nuo visų judamosios apkrovos padėčių $\xi \in P$. Esant nesimetrinei santvarai, ekstreminių jėgų vektoriams $\mathbf{N}_{e,max}$ ir $\mathbf{N}_{e,min}$ rasti reikia papildomai sudaryti dar penkis vektorius $\mathbf{F}_\xi$, kai jėgos V_1 ir V_2 sukeistos vietomis t. y.: $\hat{\mathbf{F}}_6 = (V_1,\ 0,\ 0,\ 0)^T$, $\hat{\mathbf{F}}_7 = (V_2,\ V_1,\ 0,\ 0)^T$, $\hat{\mathbf{F}}_8 = (0,\ V_2,\ V_1,\ 0)^T$, $\hat{\mathbf{F}}_9 = (0,\ 0,\ V_2,\ V_1)^T$, $\hat{\mathbf{F}}_{10} = (0,\ 0,\ 0,\ V_2)^T$. Santvaros tūrio minimizavimo uždavinio (1a)–(1i) sprendimo metu $\mathbf{N}_{e,max}$ ir $\mathbf{N}_{e,min}$ kinta, nes priklauso nuo santvaros fizikinių ir geometrinių parametrų. Patikrinamajame uždavinyje (2a)–(2i) $\mathbf{N}_{e,max}$ ir $\mathbf{N}_{e,min}$ priklauso tik nuo apkrovos kitimo ribų (α šiuo atveju nesikeičia).

3. Naujas prisitaikančiųjų santvarų optimizavimo uždavinių sprendimo algoritmas

3.1. Moro integralo interpretacija plastinių konstrukcijų analizėje

Pagrindinio netiesinio santvaros minimalaus tūrio uždavinio (1a)–(1i) tiesioginis sprendimas yra gana sudėtingas, nes sprendimo metu keičiasi santvaros strypų standžiai EA_j, $j \in J$ (visos santvaros tamprumo modulis laikomas pastoviu). Tai reiškia, kad keičiasi influentinės $[\alpha]$, $[\beta]$, $[G]$ ir $[H]$ matricos. Nemažus sprendimo sunkumus savo ruožtu sukelia ir matematinio programavimo griežtumo sąlygos (1g) ir standumo sąlygų (1i) (arba (2g)–(2i)) tikrinimas. Standumo sąlygos įvedamos dėl to, kad KKA atveju įmanomas skerspjūvių, esančių plastinės stadijos, nusikrovimas. Jeigu nenagrinėjama apkrovimo istorija, tai uždavinio (1a)–(1i) „viduje“ tenka spręsti tiesinius uždavinius, nustatant: $\mathbf{u}_{r,inf} = \min[H]\mathbf{\Theta}_p$, $\mathbf{u}_{r,sup} = \max[H]\mathbf{\Theta}_p$. Jeigu pradiniais sprendimo etapais nusikrovimas ignoruojamas, sąlyga (1i) (arba (2i)) užrašoma taip:

$$\mathbf{u}_{r,\min} \le [H]\mathbf{\Theta}_p \le \mathbf{u}_{r,\max}. \tag{5}$$

Sąlyga (2c) iš esmės išreiškia Moro integralą tampriai plastinei sistemai. Tegul ribojamas *i*-tasis liekamasis poslinkis $u_{r,i}$:

$$u_{r,i} = \sum \int_l \frac{N_r^* \overline{N}_{er,i}}{EA} = \mathbf{N}_r^{*T}[D]\overline{\mathbf{N}}_{er},\ i \in I, \tag{6}$$

čia N_r^* – analizės uždavinio, kuris figūruoja uždaviniuose (1a)–(1i), (2a)–(2i) optimalus sprendinys, $\overline{\mathbf{N}}_{er}$ – santvaros strypų ašinės jėgos nuo vienetinės jėgos $\overline{F}_i = 1{,}0$ ($\overline{\boldsymbol{N}}_{er}$ skaičiuojama santvaroje, atsižvelgus į jos statiško neišsprendžiamumo laipsnio sumažėjimą, vystantis plastinėms deformacijoms).

3.2. Etapinis minimalaus tūrio santvaros uždavinio sprendimo algoritmas

Aptartos standumo sąlygos (5), (6) pagrindiniame optimizavimo uždavinyje (1a)–(1i) pakeičiamas trijų tarpinių uždavinių sprendimu.

Pirmasis tarpinis uždavinys. Pasirinkus santvaros strypų skerspjūvius A_j, $j \in J$, formuojamos matricos $[\alpha]$, $[\beta]$, $[G]$ ir $[H]$. Kadangi žinomas jėgų sistemos dydis ir jos pridėjimo padėtys, pagal formules (4) apskaičiuojami $\mathbf{N}_{e,\max} = (\mathbf{N}_{e1,\max}, \ \mathbf{N}_{e2,\max}, \ \ldots, \ \mathbf{N}_{en,\max})^T$ ir $\mathbf{N}_{e,\min} = (\mathbf{N}_{e1,\min}, \ \mathbf{N}_{e2,\min}, \ \ldots, \ \mathbf{N}_{en,\min})^T$ komponentai.

Antrasis tarpinis uždavinys. Analizės uždavinio sprendimas:
rasti

$$\min \frac{1}{2} \mathbf{N}_r^T [D] \mathbf{N}_r \ , \tag{7a}$$

kai

$$\mathbf{f}_{\max} = \mathbf{N}_0 - [G]\boldsymbol{\Theta}_p - \mathbf{N}_{e,\max} \geq \mathbf{0} \, , \tag{7b}$$

$$\mathbf{f}_{\min} = \mathbf{N}_{0,cr} + [G]\boldsymbol{\Theta}_p + \mathbf{N}_{e,\min} \geq \mathbf{0} \, , \tag{7c}$$

Sprendžiant šį uždavinį naudojamasi pirmojo tarpinio uždavinio sprendimo rezultatais, būtent vektoriais $\mathbf{N}_{e,\max}$ ir $\mathbf{N}_{e,\min}$. Antrojo tarpinio uždavinio sprendimo rezultatai yra $\mathbf{N}_r^*$, $\mathbf{u}_r^*$, Θ_p^*. Gavus $\mathbf{u}_r^*$, iš dalies galima pasitikrinti (5) sąlygą. Nė viena iš tų sąlygų neturėtų būti pažeista. Priešingu atveju didinami santvaros strypų skerspjūvio plotai A_j, $j \in J$ ir grįžtama prie pirmojo tarpinio uždavinio.

Turint antrojo tarpinio uždavinio sprendinį $\mathbf{N}_r^*$ ir žinant $\mathbf{N}_{e,\max}$ ir $\mathbf{N}_{e,\min}$ skaičiuojami suminiai $\mathbf{N}_{\max} = \mathbf{N}_r^* + \mathbf{N}_{e,\max}$, $\mathbf{N}_{\min} = \mathbf{N}_r^* + \mathbf{N}_{e,\min}$ ir sudaromas vektorius. Šie vektoriai – $\mathbf{N}_r^*$, $\mathbf{N}_{e,\max}$, $\mathbf{N}_{e,\min}$, $\overline{\mathbf{N}}_{er}$ yra pradiniai duomenys trečiajam tarpiniam uždaviniui spręsti.

Trečiasis tarpinis uždavinys. Šio uždavinio matematinis modelis toks:
rasti

$$\min \sum_j L_j A_j \, , \tag{8a}$$

kai

$$\sigma_y A_j \geq \left(N_{rj}^* + N_{ej,\max} \right),$$

$$\varphi_j \sigma_y A_j \geq \left(N_{rj}^* + N_{ej,\min} \right), \tag{8b}$$

$$u_{ri,\min} \leq \sum_l \int \frac{N_r^* \overline{N}_{er,i}}{EA} \leq u_{ri,\max} \, , \tag{8c}$$

$$u_{ri,\min} \leq \sum_l \int \frac{N_r^* \overline{N}_{er,i}}{EA} \leq u_{ri,\max} \, . \tag{8d}$$

Šiame uždavinyje nežinomieji yra strypų skerspjūvio plotai A_j, $j \in J$. Tai iškilojo programavimo uždavinys.

3.3. Liekamųjų poslinkių analitinės išraiškos

Šis algoritmas panašus į aprašytąjį 4.2 skyriuje. Skiriasi tik standumo apribojimų (3) išraiška, kuri užrašoma taip:

$$\mathbf{u}_{r,\min} \leq [H]\boldsymbol{\Theta}_p \leq \mathbf{u}_{r,\max} \, , \tag{9}$$

čia matricos $[H]$ komponentai yra analitinės išraiškos, gautos panaudojus kompiuterinės algebros paketą MAPLE. Analitinės išraiškos leidžia lengvai suskaičiuoti ir apribojimų (9) gradientus, kurie reikalingi sprendžiant uždavinį (1a)–(1i) Rozeno projektuojamųjų gradientų metodu [18]. Sprendimo etapai analogiški 4.2 skyriaus optimizavimo uždaviniui. Tačiau trečiasis tarpinis uždavinys užrašomas taip:
rasti

$$\min \sum_j L_j A_j \, , \tag{10a}$$

kai

$$\sigma_y A_j \geq \left(N_r^* + N_{e,\max} \right),$$

$$\varphi_j \sigma_y A_j \geq \left(N_r^* + N_{e,\min} \right), \tag{10b}$$

$$\mathbf{u}_{r,\min} \leq [H]\boldsymbol{\Theta}_p \leq \mathbf{u}_{r,\max} \, , \tag{10c}$$

$$A_j \geq A_{j,\min} \, . \tag{10d}$$

Reikėtų pažymėti, jog analitinių matricos $[H]$ bei apribojimų (10c) išraiškų formavimas yra imlus kompiuterio resursų procesas.

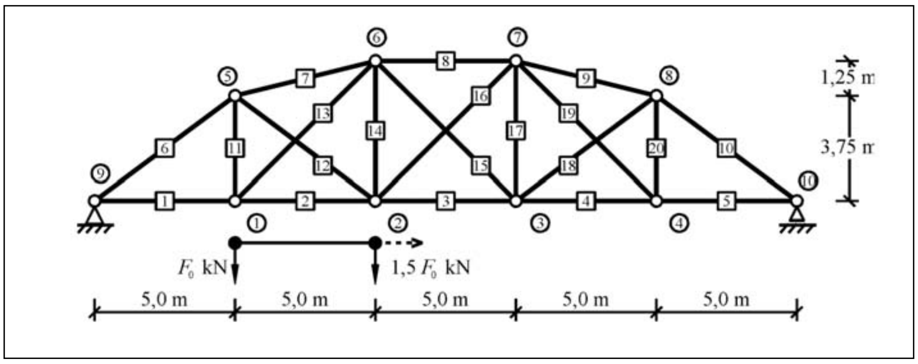
4. Pavyzdžiai

4.1 pavyzdys. Nagrinėjama tiltinė santvara (2 pav.) apkrauta dviejų judančiųjų jėgų sistemos: $0 \leq F_1 \leq F_{1,sup} = 370$ kN ir $0 \leq F_2 \leq F_{2,sup} = 555$ kN.

Santvara sudaryta iš 20 strypų, jos laisvės laipsnis 17. Tamprumo modulis $E = 21\,000$ kN/cm^2, visų strypų takumo riba $\sigma_y = 20$ kN/cm^2. Santvaros strypai grupuojami į keturias grupes (viršutinės juostos, apatinės juostos, vertikalūs, įstriži), kur kiekvienos grupės strypų skerspjūvio plotai yra lygūs. Parinkti minimalūs skerspjūvių plotai yra:

$$A_{bot,\min} = A_{top,\min} = A_{diag,\min} = A_{vert,\min} = 10 \, \text{cm}^2.$$

Pagrindinė užduotis – išspręsti tūrio minimizacijos uždavinį (1a)–(1i) t. y. rasti skerspjūvio plotus A_k, $k = 1, 2, \ldots, 20$ atitinkančius kriterijų (1a) $\min \sum_j L_j A_j$, šiais atvejais:



2 pav. Santvaros geometrija bei apkrova

Fig 2. Geometry and load of the truss

A1, kai įvertintos stiprumo (1b)–(1c) ir standumo (1i) sąlygos;

A2, kai įvertintos visos – stiprumo, standumo ir stabilumo – sąlygos

Standumo apribojimai realizuojami naudojant vertikaliųjų mazgų poslinkių suvaržymus $|u_i| \le 3$ cm.

Tūrio minimizacijos uždavinys (1a–1i) spręstas iteracijomis (sprendimo metu kinta $\mathbf{N}_{e,\max}$ ir $\mathbf{N}_{e,\min}$, nes kinta santvaros fizikiniai ir geometriniai parametrai). A1 atveju gautas minimalus santvaros tūris $V_{min} = 471\ 710$ cm³, o A2 – $V_{min} = 569\ 100$ cm³.

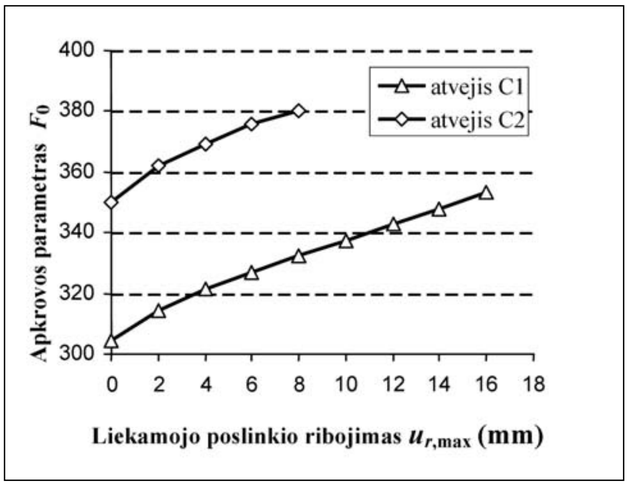
4.2 pavyzdys. Nagrinėjama dvidešimties strypų (jų skerspjūvis žiedinės formos) santvara (2 pav.), veikiama judamosios apkrovos (dviejų jėgų sistemos, kurių pirmoji – $1{,}5F_0$, antroji – F_0). Santvaros strypų medžiagos tamprumo modulis $E = 21000$ kN/cm² ir takumo riba $\sigma_y = 20$ kN/cm². Strypų skerspjūvio plotai A_k bei klupumo koeficientai φ_k, $k = 1, 2, ..., 20$ yra tokie:
$A_1 = A_2 = A_3 = A_4 = A_5 = A_{12} = A_{13} = A_{15} = A_{16} = A_{18} = 48{,}25$ cm², $A_6 = A_{10} = 72{,}26$ cm², $A_7 = A_8 = A_9 = 46{,}84$ cm², $A_{11} = A_{14} = A_{17} = A_{20} = 43{,}23$ cm², $A_{19} = 48{,}25$ cm², $\varphi_1 = \varphi_2 = \varphi_3 = \varphi_4 = \varphi_5 = 0{,}837$, $\varphi_6 = \varphi_{10} = 0{,}821$, $\varphi_7 = \varphi_9 = 0{,}860$, $\varphi_8 = 0{,}869$, $\varphi_{11} = \varphi_{20} = 0{,}889$, $\varphi_{12} = \varphi_{18} = 0{,}731$, $\varphi_{\varphi 3} = \varphi_{15} = \varphi_{16} = \varphi_{19} = 0{,}650$, $\varphi_{14} = \varphi_{17} = 0{,}792$.

Pagrindinė užduotis – rasti judamosios apkrovos maksimalią parametro F_0 reikšmę:

C1 – kai atsižvelgiama tik į stiprumo (2b)–(2c) ir standumo (2i) reikalavimus;

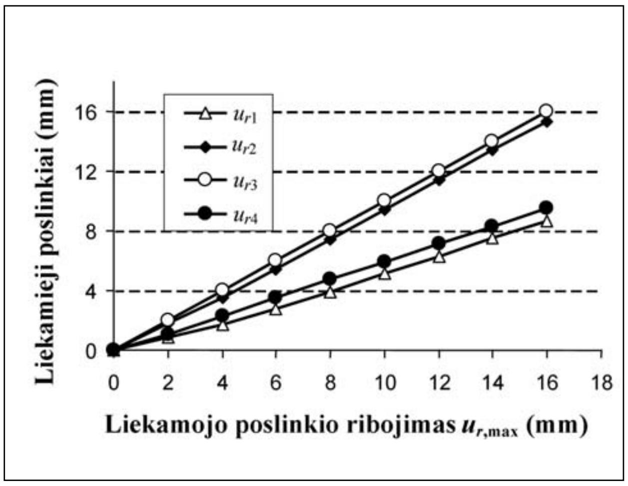
C2 – kai visos – tiek stiprumo, tiek standumo, tiek stabilumo – sąlygos įvertinamos.

Standumo sąlygos realizuojamos ribojant santvaros apatinės juostos mazgų vertikaliuosius poslinkius, įvedant skirtingas $u_{r,\max}$ reikšmes ($0 \le u_{ri} \le u_{r,\max}$, $i = 1, 2, 3, 4$). Prisitaikomumo būvio apkrovos daugiklio F_0 kitimas, esant skirtingiems $u_{r,\max}$, parodytas 3 pav. Santvaros apatinės juostos vertikaliųjų mazginių poslinkių reikšmės (C2 atveju) parodytos 4 pav. (indeksas ties u_r atitinka mazgo numerį).



3 pav. F_0 priklausomybė nuo $u_{r,\max}$

Fig 3. F_0 dependence on $u_{r,\max}$



4 pav. u_r priklausomybė nuo $u_{r,\max}$

Fig 4. u_r dependence on $u_{r,\max}$

5. Išvados

Idealiai tampriai plastinei santvarai, veikiamai judamosios apkrovos, optimizuoti įmanoma pasitelkti prisitaikomumo teorijos metodus. Jungiamąja grandimi čia yra tamprių ekstreminių ašinių jėgų skaičiavimas pagal visas apkrovų hodografo viršūnes. Apkrovos gali būti charakterizuojamos ir viršutinėmis nuo laiko nepriklausančiomis jėgų kitimo ribomis (apkrovų judėjimo tvarka tampa neaktuali). Sudarytieji nauji netiesiniai apkrovos optimizavimo ar minimalaus tūrio santvaros uždavinių matematiniai modeliai tuomet „dirba“ į atsargos pusę (negalima pasiekti realios konstrukcijos cikliškai-plastinio suirimo būvio). Skaitiniai straipsnio eksperimentai parodė ne tik siūlomų naujų sprendimo algoritmų efektyvumą, bet ir pačių optimizavimo uždavinių matematinių modelių sudarymo pagrįstumą.

Literatūra

1. ČYRAS, A. *Analysis and Optimization of Elastoplastic Systems.* John Wiley &Sons, New York, 1983. 112 p.
2. BAZANT, Z. *Inelastic Analysis of Structures in Civil Engineering.* John Wiley & Sons, New York, 1999.
3. ROZVANY, G. I. N. *Optimal design of flexural systems*. Oxford: Pergamon Press, 1976.
4. BORKOWSKI, A.; JENDO, S.; REITMAN. *Mathematical Programming*, Vol 2 of the series „Structural Optimization“. Ed. by Save, M. and Prager, W. Plenum Press, New York, 1990.
5. GIAMBANCO, F.; PALIZZOLO, L.; POLIZZOTTO, C. Optimal shakedown design of beam structures. *Structural Optimization*, 1994, Vol 8, p. 156–167.
6. KALISZKY, S.; LÓGÓ, J. Plastic behaviour and stability constraints in shakedown analysis and optimal design. *Struct. Multidisc. Optim.,* 2002, 24, p. 118–124.
7. CASCIARO, R.; GARCEA, G. An iterative method for shakedown analysis. *Comput. Methods Appl. Mech. Engrg* 191, 2002, p. 5761–5792.
8. SMITH, D. LLOYD. *CISM, Mathematical programming methods in structural plasticity*. Wien-New York: Springer-Verlag, 1990.
9. CHOI, S. H.; KIM, S. E. Optimal design of steel frame using practical nonlinear inelastic analysis. *Engineering Structures*, 2002, Vol 24 (9), p. 1189–1201.
10. ATKOČIŪNAS, J.; MERKEVIČIŪTĖ, D. Optimal Shakedown Design of Bar Systems: Strength, Stiffness and Stability Constraints. In *Proceedings of the Seventh International Conference on Computational Structures Technology*, September 7–9, 2004, Lisbon, Portugal (Eds. B. H. V. Topping and C. A. Mota Soares). Civil-Comp Press, Stirling, Scotland, 2004, p. 361–363. ISBN 0-948749-93-8.
11. DAPŠEVIČIŪTĖ, I.; ATKOČIŪNAS, J. Prisitaikančių santvarų optimizacija: judamosios apkrovos atvejis. Iš *7-osios Lietuvos jaunųjų mokslininkų konferencijos „Lietuva be mokslo – Lietuva be ateities“, įvykusios Vilniuje 2004 m. kovo 25–26 d., medžiaga. Statyba.* Vilnius: Technika, 2004, p. 277–282. ISBN 9986-05-893-7.
12. CEN, EN 1993-1-1, Eurocode 3: Design of Steel Structures, Part 1.1: General Rules and Rules for Buildings, 4th Draft, Brussels, October 2001.
13. FERRIS, M. C.; TIN-LOI, F. On the solution of a minimum weight elastoplastic problem involving displacement and complementarity constraints. *Computer Methods in Applied Mechanics and Engineering*, 1999, 174, p. 107–120.
14. VENSKUS, A.; ATKOČIŪNAS, J. Patobulintas prisitaikančių sistemų optimizacijos uždavinių sprendimo algoritmas. Iš *STATYBA. 9-osios Lietuvos jaunųjų mokslininkų konferencijos „Mokslas – Lietuvos ateitis“, įvykusios Vilniuje 2006 m. kovo 29–31 d., pranešimų rinkinys.* Vilnius: Technika, 2006, p. 265–270. ISBN 9955-28-047-6.
15. Staat, M.; Heitzer, M. (eds.). *Numerical methods for limit and shakedown analysis. Series of John von Neumann Institute for Computing*, Vol 15, 2003.
16. MERKEVIČIŪTĖ, D.; ATKOČIŪNAS, J. Minimum volume of trusses at shakedown – mathematical models and new solution algorithms. *Mechanika*, 2005, Nr. 2(52), p. 47–54. ISSN 1392-1207.
17. ATKOČIŪNAS, J. Mathematical models of optimization problems at shakedown. *Mech. Res. Commun.,* 1999, Vol 26, No 3, p. 319–326.
18. BAZARAA, M. S.; SHERALI, H. D.; SHETTY, C. M. *Nonlinear programming: theory and algorithms*. New York: Brijbasi Art Press Ltd., John Wiley & Sons, Inc., 2004. 638 p.

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ISSN 1392 - 1207. MECHANIKA. 2007. Nr.2(64)

Nonlinear programming and optimal shakedown design of frames

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1. Introduction

Steel frames, which undergo plastic strains and are subjected to variable repeated load, are considered in the paper. Under repeated loading a structure can lose its serviceability because of its progressive plastic failure or because of alternating strain (usually both cases are called cyclic-plastic collapse). The third case when the structure adapts to the existing load and further behaves only elastically is also possible. For civil engineering, the calculation of any complexity elastic–plastic frames subjected to variable repeated load is relevant. Growing number of scientific works dedicated to adapted structure calculation shows importance of these researches [1 - 8]. But there is especially small number of works concerning the optimization of adapted structures under stiffness constraints. This had an influence on the topic of this paper: optimal shakedown design of frames, subjected to variable repeated load, under stiffness constraints. Herein two types of problems can be considered [9]. The first problem is optimal shakedown design of cross-sectional parameters (design problem) and the second one - load optimization problem for a frame subjected to variable repeated load (checking problem). By solving checking problem maximal load variation bounds, ensuring adapted state of the frame and satisfying stiffness requirements of the structure, are to be found.

Solution of frame optimization problems at shakedown is complicated as stress–strain state of dissipative systems depends on loading history [10 - 14]. These difficult optimization problems are implemented applying extremum energy principles and the theory of mathematical programming [15]. That enables to create new iterative algorithm based on Rosen project gradient method [16-19]. Numerical examples of the frames are presented. The results are valid for small displacement assumptions.

2. General mathematical models of optimization problems at shakedown

General mathematical models presented in Table are the basis for the development of optimization mathematical models of frames at shakedown considered in this paper. In both design and checking problems objective functions are described by formulas (1) and (6), where the vectors $\boldsymbol{L}$, $\boldsymbol{T}_{sup}$ and $\boldsymbol{T}_{inf}$ contain coefficients of weight. Yield conditions $\boldsymbol{\varphi}_j$ ($j \in J$) are shown in formulas (2) and (7), where j is the number of all possible combinations $\boldsymbol{F}_j$ of load bounds $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$. Formulas (3) and (8) represent complementary slackness conditions of mathematical programming, (4) and (9) are constraints for the problem unknowns. Stiffness constraints are shown in (5) and (10).

Discrete model of the frame at shakedown consists of s ($k = 1, 2, ..., s$, $k \in K$) finite elements. Limit force S_{0k} ($k \in K$) is assumed as constant in the whole finite element. The degree of freedom is m, corresponding m - vector of displacements - $\boldsymbol{u}_e = \left(u_{e,1}, u_{e,2}, ..., u_{e,m}\right)^T$. Nodal internal forces of the element compound one n – vector of discrete model forces $\boldsymbol{S} = \left(\boldsymbol{S}_1, \boldsymbol{S}_2, ..., \boldsymbol{S}_v, ..., \boldsymbol{S}_\varsigma\right)^T = \left(S_z\right)^T$ and strains – n –vector $\boldsymbol{\Theta} = \left(\boldsymbol{\Theta}_1, \boldsymbol{\Theta}_2, ..., \boldsymbol{\Theta}_v, ..., \Theta_\varsigma\right)^T = \left(\Theta_z\right)^T$,

Table

General mathematical models of optimization problems

Design problem		Checking problem	
find		find	
$min\ \psi\left(\boldsymbol{S}_0\right) = min\ \boldsymbol{L}^T \boldsymbol{S}_0$	(1)	$max\ \left(\boldsymbol{T}_{sup}^T \boldsymbol{F}_{sup} + \boldsymbol{T}_{inf}^T \boldsymbol{F}_{inf}\right)$	(6)
subject to		subject to	
$\boldsymbol{\varphi}_j = \boldsymbol{S}_0 - \boldsymbol{\Phi}\left(\boldsymbol{G\lambda} + \boldsymbol{S}_{ej}\right) \geq \boldsymbol{0}$	(2)	$\boldsymbol{\varphi}_j = \boldsymbol{S}_0 - \boldsymbol{\Phi}\left(\boldsymbol{G\lambda} + \boldsymbol{S}_{ej}\right) \geq \boldsymbol{0}$	(7)
$\boldsymbol{\lambda}_j^T \boldsymbol{\varphi}_j = 0,\ \boldsymbol{\lambda}_j \geq \boldsymbol{0}$ $\boldsymbol{\lambda} = \sum_j \boldsymbol{\lambda}_j,\ j \in J$	(3)	$\boldsymbol{\lambda}_j^T \boldsymbol{\varphi}_j = 0,\ \boldsymbol{\lambda}_j \geq \boldsymbol{0}$ $\boldsymbol{\lambda} = \sum_j \boldsymbol{\lambda}_j,\ j \in J$	(8)
$\boldsymbol{S}_0 \geq 0$	(4)	$\boldsymbol{F}_{sup} \geq 0,\quad \boldsymbol{F}_{inf} \geq 0$	(9)
$\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf},\quad \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max}$	(5)	$\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf},\quad \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max}$	(10)

$v = 1, 2, \ldots, \zeta$ ($v \in Z$), $z = 1, 2, \ldots, n$. The total number of design sections is ζ.

Load $\boldsymbol{F}(t)$ is characterized by time t, independent variation bounds $\boldsymbol{F}_{sup} = (F_{1,sup}, F_{2,sup}, \ldots, F_{m,sup})^T$ and $\boldsymbol{F}_{inf} = (F_{1,inf}, F_{2,inf}, \ldots, F_{m,inf})^T$ ($\boldsymbol{F}_{inf} \le \boldsymbol{F}(t) \le \boldsymbol{F}_{sup}$). Elastic displacements $\boldsymbol{u}_e(t)$ and forces $\boldsymbol{S}_e(t)$ of the structure are determined using influence matrixes of displacements and forces, $\boldsymbol{\beta} = (\boldsymbol{A K A}^T)^{-1}$, $\boldsymbol{\alpha} = \boldsymbol{K A}^T \boldsymbol{\beta}$, respectively: $\boldsymbol{u}_e(t) = \boldsymbol{\beta} \boldsymbol{F}(t)$, $\boldsymbol{S}_e(t) = \boldsymbol{\alpha} \boldsymbol{F}(t)$, $\boldsymbol{K} = \boldsymbol{D}^{-1}$. Here $\boldsymbol{A}$ is a coefficient matrix of equilibrium equations $\boldsymbol{A S} = \boldsymbol{F}$ and $\boldsymbol{D}$ is a quasi-diagonal flexibility matrix. Residual displacements $\boldsymbol{u}_r$ and forces $\boldsymbol{S}_r$ are related to the vector of plasticity multipliers $\boldsymbol{\lambda}$ by influence matrixes $\boldsymbol{H}$ and $\boldsymbol{G}$: $\boldsymbol{u}_r = \bar{\boldsymbol{H}} \boldsymbol{\Phi}^T \boldsymbol{\lambda} = \boldsymbol{H} \boldsymbol{\lambda}$, $\boldsymbol{S}_r = \bar{\boldsymbol{G}} \boldsymbol{\Phi}^T \boldsymbol{\lambda} = \boldsymbol{G} \boldsymbol{\lambda}$, $\bar{\boldsymbol{H}} = (\boldsymbol{AKA}^T)^{-1} \boldsymbol{AK}$ and $\bar{\boldsymbol{G}} = \boldsymbol{KA}^T \bar{\boldsymbol{H}} - \boldsymbol{K}$. Here $\boldsymbol{\Phi}$ – the matrix of peace-wise linearized yield conditions $\boldsymbol{\varphi}_j$ (2) and (7). The number of all possible combinations $\boldsymbol{F}_j$ of load bounds $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$ is $p = 2^m$ ($\boldsymbol{F}_{inf} \le \boldsymbol{F}_j \le \boldsymbol{F}_{sup}$): $\boldsymbol{S}_{ej} = \boldsymbol{\alpha} \boldsymbol{F}_j$, $j = 1, 2, \ldots, p$, ($j \in J$). In the case of two loads F_1, F_2, a domain of elastic force variation (locus) is shown in Fig. 1.

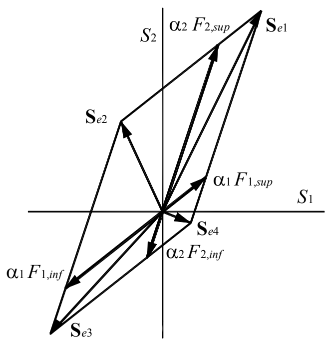


Fig. 1 Locus of elastic forces

Residual displacements $\boldsymbol{u}_r$ of the structure at shakedown can be nonunique: they depend on particular loading history $\boldsymbol{F}(t)$. If load is defined only by variation bounds $\boldsymbol{F}_{inf}$, $\boldsymbol{F}_{sup}$, the calculation of exact values of residual displacements becomes problematical because of unloading phenomenon appearing at cross-sections: then displacements $\boldsymbol{u}_r$ are varying nonmonotonically, it is possible to determine only their lower $\boldsymbol{u}_{r,inf}$ and upper $\boldsymbol{u}_{r,sup}$ variation bounds ($\boldsymbol{u}_{r,inf} \le \boldsymbol{u}_r(t) \le \boldsymbol{u}_{r,sup}$). Stiffness conditions (5) and (10) are realized by the restriction of the structure nodal displacement lower and upper variation bounds $\boldsymbol{u}_{r,min} \le \boldsymbol{u}_{r,inf}$, $\boldsymbol{u}_{r,sup} \le \boldsymbol{u}_{r,max}$.

Mathematical programming theory, the widely used method of the solution of extremum problems, helps not only for the formulation of shakedown problems theory, but also for its solution. Problems (1)-(5) and (6)-(10) can be solved by various computer programs but in this case mechanical interpretation possibilities of optimality criterion of applied algorithms are not revealed. In our works mechanical interpretation of optimality conditions for Rozen algorithm is revealed – it is strain compatibility equations [20].

3. Rozen project gradient method

Rosen project gradient algorithm is universal enough, that it can be applied when objective function and constraints are linear (1) - (5), (6) - (10), or nonlinear [20]. For the optimization problems of volume minimization and determination of maximal load variation bounds containing linear objective function and constraints, application of the Rosen algorithm will be shown. Generally the convex problem of linear programming reads
find

$$max\ \mathscr{F}(\boldsymbol{x}) \tag{11}$$

subject to

$$\varphi_i(\boldsymbol{x}) = \boldsymbol{a}_i^T \boldsymbol{x} \le 0\,,\ i = 1, 2, \ldots, l\,,\ i \in I \tag{12}$$

As function $\varphi_i(\boldsymbol{x})$ is linear, its gradient is $\nabla\varphi_i(\boldsymbol{x}) = \boldsymbol{a}_i$; here $\boldsymbol{a}_i$ is n-vector of multipliers near unknown quantities. In the case of linear constraints (12) gradient matrix of active constraints is noted $\boldsymbol{A}_\kappa$ i.e.

$$\nabla\boldsymbol{\Phi}(\boldsymbol{x}) = \boldsymbol{A}_\kappa = [\boldsymbol{a}_1\ \boldsymbol{a}_2\ \ldots\ \boldsymbol{a}_i\ \ldots\ \boldsymbol{a}_\kappa] \tag{13}$$

here $\boldsymbol{A}_\kappa$ is $(n \times \kappa)$ – order unit matrix, where n is the measure of Euclidian space E^n and κ is the number of active constraints. Constraints, which are satisfied as equalities, ($\varphi_i(\boldsymbol{x}^k) = 0$, $i \in I$) are called active ones. Vectors from n-dimensional space, satisfying conditions (12) as equalities, compound $(n \times \kappa)$-order formation noted as G^κ. In Euclidian space E^n movement from $\boldsymbol{x}^k$ is performed in the direction of vector $\boldsymbol{P}_\kappa \nabla\mathscr{F}(\boldsymbol{x}^k)$ (Fig. 2), which is calculated according to the formula

$$\boldsymbol{P}_\kappa \nabla\mathscr{F}(\boldsymbol{x}^k) = \left(\boldsymbol{I} - \nabla\boldsymbol{\Phi}(\boldsymbol{x}^k) \boldsymbol{V}_\kappa(\boldsymbol{x}^k) \nabla\boldsymbol{\Phi}^T(\boldsymbol{x}^k)\right) \nabla\mathscr{F}(\boldsymbol{x}^k) \tag{14}$$

$\boldsymbol{I}$ is $(n \times n)$-order unit matrix, $\nabla\mathscr{F}(\boldsymbol{x}^k)$ is the gradient of objective function and $(\kappa \times \kappa)$-matrix $\boldsymbol{V}_x(\boldsymbol{x}^k)$ is expressed as follows: $\boldsymbol{V}_x(\boldsymbol{x}^k) = \left(\nabla\boldsymbol{\Phi}^T(\boldsymbol{x}^k) \nabla\boldsymbol{\Phi}(\boldsymbol{x}^k)\right)^{-1}$. $\boldsymbol{P}_\kappa$ is a projective matrix.

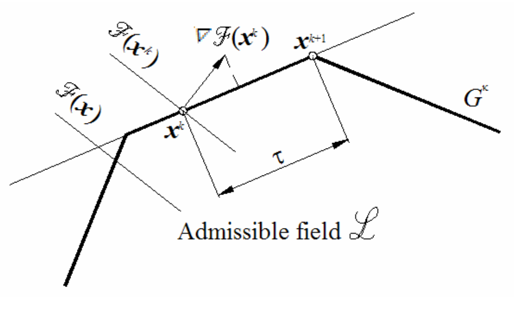


Fig. 2 Rosen algorithm for linear constraints

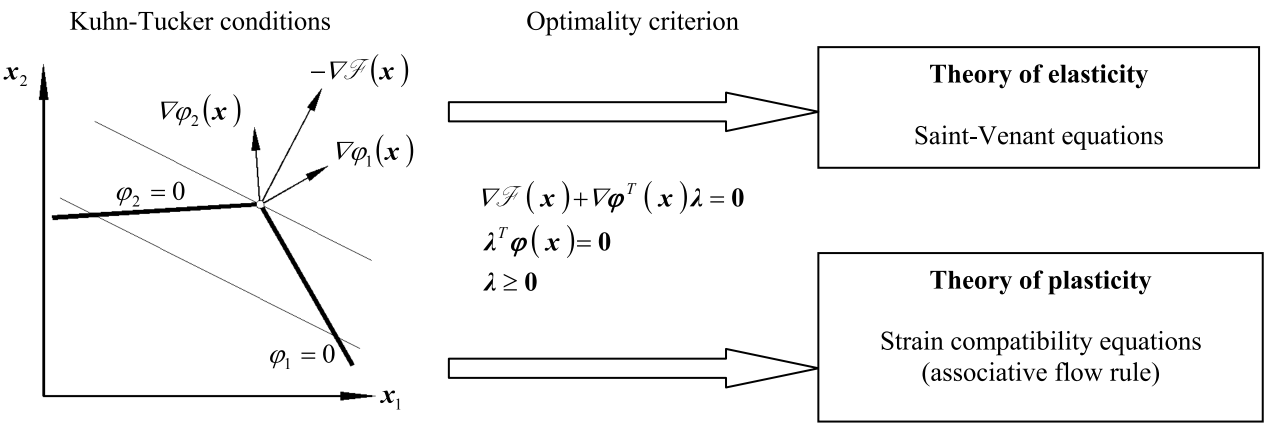


Fig. 3 Kuhn-Tucker optimality conditions are strain compatibility equations of the deformable body mechanic

Vector $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \tau' \boldsymbol{P}_\kappa \nabla\mathscr{F}(\boldsymbol{x}^k)$, where $\tau' = min\ \{\tau' | \tau' > 0,\ \ i = \kappa+1,\ \kappa+2,...,l\}$ is the step of the move. Only so vector $\boldsymbol{x}^{k+1}$ "does not leave" admissible field $\mathscr{L} = \{\boldsymbol{x} \,|\, \varphi_i(\boldsymbol{x}) \leq 0,\ i = 1,2,...,l\}$. If the vector does not exist in the admissible range $0 < \tau < \tau'$, for which the magnitude of objective function would be greater than at point $\boldsymbol{x}^{k+1}$ then it is assumed that $\tilde{\boldsymbol{x}}^{k+1} = \boldsymbol{x}^{k+1}$ and the calculation process is continued. If $\nabla\mathscr{F}^T(\boldsymbol{x}^k)\boldsymbol{P}_\kappa\nabla\mathscr{F}(\boldsymbol{x}^{k+1}) < 0$, then the objective function reaches its maximum in the radius between points $\boldsymbol{x}^k$ and $\boldsymbol{x}^{k+1}$. The new size of the step is calculated as follows

$$\tau'' = \tau' \frac{\nabla\mathscr{F}^T(\boldsymbol{x}^k)\boldsymbol{P}_\kappa\nabla\mathscr{F}(\boldsymbol{x}^k)}{\nabla\mathscr{F}^T(\boldsymbol{x}^k)\boldsymbol{P}_\kappa\nabla\mathscr{F}(\boldsymbol{x}^k) - \nabla\mathscr{F}(\boldsymbol{x}^k)\boldsymbol{P}_\kappa\nabla\mathscr{F}(\boldsymbol{x}^{k+1})} \quad (15)$$

In this case $\boldsymbol{x}^{k+1}$ is determined according to the formula: $\boldsymbol{x}^{k+1} = \boldsymbol{x}^k + \tau'' \boldsymbol{P}_\kappa \nabla\mathscr{F}(\boldsymbol{x}^k)$. Vector $\boldsymbol{x}$ is the solution if the following conditions are satisfied

$$\boldsymbol{P}_\kappa \nabla\mathscr{F}(\boldsymbol{x}) = \mathbf{0}, \quad (16)$$

$$\boldsymbol{V}_x(\boldsymbol{x})\nabla\boldsymbol{\Phi}^T(\boldsymbol{x})\nabla\mathscr{F}(\boldsymbol{x}) \leq \mathbf{0} \quad (17)$$

For correct mechanical interpretation of the conditions (16), it falls to use Kuhn-Tucker conditions [17]. So it is done in the research [20], where it is shown that equation (16) is strain compatibility equation (Fig. 3) and the left side of inequality (17) in absolute value is a vector of plastic multipliers $\boldsymbol{\lambda}$

$$\boldsymbol{\lambda} = \left|\boldsymbol{V}_x(\boldsymbol{x})\nabla\boldsymbol{\Phi}^T(\boldsymbol{x})\nabla\mathscr{F}(\boldsymbol{x})\right| \quad (18)$$

4. Design of minimal volume frame at shakedown

Design of the frame for optimal parameters is performed when yield limit σ_{yk} of the frame material and lengths L_k of its all elements k ($k \in K$) and load variation bounds $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$ are known. Depending on the cross-sectional shape various yield conditions can be considered. In this paper, the focus is placed on yield conditions for rolled I steel sections (Fig. 4). Relation $c_k = \dfrac{M_{0k}}{N_{0k}}$, $k \in K$ should be prescribed in advance. Limit moment $M_{0k} = \sigma_{yk} W_{pl,k} = \xi(\sigma_{yk}, A_k)$ and limit axial force $N_{0k} = \sigma_{yk} A_k$ of the element are functions of cross-sectional area A_k and yield limit of material σ_{yk}. True, usually one or the other specific dimension of the cross-section (for instance, flange thickness t_f of I-section while the width of flange b is fixed; see Example 1) participate in functional relation $M_{0k} = \xi(\sigma_{yk}, A_k)$ instead of cross-sectional area A_k. The problem of frame optimal parameters distribution design reads: minimize $\sum_k L_k M_{0k}$, subject to the structure strength and stiffness constraints find

$$min \ \sum_k L_k M_{0k} \quad (19)$$

subject to

$$\boldsymbol{\varphi}_j = \boldsymbol{M}_0 - \boldsymbol{\Phi}(\boldsymbol{G}\boldsymbol{\lambda} + \boldsymbol{S}_{ej}) \geq \mathbf{0} \quad (20)$$

$$\sum_j \boldsymbol{\lambda}_j^T \boldsymbol{\varphi}_j = 0,\ \boldsymbol{\lambda}_j \geq \mathbf{0},\ \boldsymbol{\lambda} = \sum_j \boldsymbol{\lambda}_j \quad (21)$$

$$M_{0k,max} \geq M_{0k} \geq M_{0k,min},\ \ k \in K,\ \ j \in J \quad (22)$$

$$\boldsymbol{u}_{min} \leq \boldsymbol{u}_{r,inf},\quad \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{max} \quad (23)$$

Limit moments M_{0k} of the frame elements and vectors of plasticity multipliers $\boldsymbol{\lambda}_j \geq \mathbf{0}$, $j \in J$ are unknowns of nonlinear mathematical programming problem (19)-(23). Formulas (21) represent complementary slackness conditions of mathematical programming [21]. Constructive requirements of frames $M_{0k,\,max}$ and $M_{0k,\,min}$ are shown in conditions (22). Problem (19)-(23) is not exactly the volume minimization problem, because limit moments M_{0k} are used in objective function. When volume of the frame is directly included into objective function mathematical model of the frame volume minimization is as follows find

$$min \sum_k L_k A_k \quad (24)$$

subject to

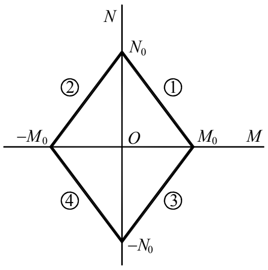
$$\boldsymbol{\varphi}_j = \boldsymbol{M}_0 - \boldsymbol{\Phi}\left(\boldsymbol{G}\,\boldsymbol{\lambda} + \boldsymbol{S}_{ej}\right) \geq \boldsymbol{0} \tag{25}$$

$$\sum_j \boldsymbol{\lambda}_j^T \boldsymbol{\varphi}_j = 0\,,\ \boldsymbol{\lambda}_j \geq \boldsymbol{0}\,,\ \boldsymbol{\lambda} = \sum_j \boldsymbol{\lambda}_j\,,\ j \in J \tag{26}$$

$$A_k \geq A_{k,min}\,,\quad k \in K \tag{27}$$

$$\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf}\,,\qquad \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max} \tag{28}$$

Cross-sectional areas A_k, $k \in K$ (or other specific dimension of the cross-section) of the frame elements and vectors of plasticity multipliers $\boldsymbol{\lambda}_j \geq \boldsymbol{0}$, $j \in J$ are unknowns of nonlinear mathematical programming problem (24)-(28).



$$\boldsymbol{\Phi}_v = \begin{bmatrix} 1 & c_k \\ -1 & c_k \\ 1 & -c_k \\ -1 & -c_k \end{bmatrix} \qquad v \in Z$$

Fig. 4 Linear yield conditions

Lower bounds of cross-sectional areas $A_{k,min}$ are included into constructive constraints (27) $A_k \geq A_{k,min}$. It is not difficult to introduce elastic displacements into stiffness constraints (28). Limit moments $\boldsymbol{M}_0$ and influence matrixes $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{G}$, $\boldsymbol{H}$ are related with unknowns A_k, $k \in K$; the listed matrixes are recalculated during solution of the problem (24)-(28). If stiffness constrains are neglected, cyclic-plastic collapse of the frame is reached.

When only bending moments M are taken in to account in the frame calculation, the following mathematical model of the frame volume minimization is obtained find

$$min \sum_k L_k A_k \tag{29}$$

subject to

$$\boldsymbol{\varphi}_{max} = \boldsymbol{M}_0 - \boldsymbol{G}\,\boldsymbol{\lambda} - \boldsymbol{M}_{e,max} \geq \boldsymbol{0}$$

$$\boldsymbol{\varphi}_{min} = \boldsymbol{M}_0 + \boldsymbol{G}\,\boldsymbol{\lambda} + \boldsymbol{M}_{e,min} \geq \boldsymbol{0} \tag{30}$$

$$\boldsymbol{\lambda}_{max}^T \boldsymbol{\varphi}_{max} = 0\,,\ \boldsymbol{\lambda}_{min}^T \boldsymbol{\varphi}_{min} = 0\,,\ \boldsymbol{\lambda}_{max} \geq \boldsymbol{0}\,,\ \boldsymbol{\lambda}_{min} \geq \boldsymbol{0} \tag{31}$$

$$\boldsymbol{\lambda} = \left(\boldsymbol{\lambda}_{\max}, \boldsymbol{\lambda}_{min}\right)^T \tag{32}$$

$$A_k \geq A_{k,min}\,,\quad k \in K \tag{33}$$

$$\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf}\,,\quad \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max} \tag{34}$$

Extreme elastic bending moments $\boldsymbol{M}_{e,\max} = \boldsymbol{\alpha}_{sup}\,\boldsymbol{F}_{sup} - \boldsymbol{\alpha}_{inf}\,\boldsymbol{F}_{inf}$, $\boldsymbol{M}_{e,\min} = \boldsymbol{\alpha}_{sup}\,\boldsymbol{F}_{inf} + \boldsymbol{\alpha}_{inf}\boldsymbol{F}_{sup}$ are known in the problem (29)-(34). Matrix $\boldsymbol{\alpha}_{sup}$ is formated in the following way: only positives values are retrieved from the influence matrix $\boldsymbol{\alpha}$, the rest components are set to zero and respectively matrix $\boldsymbol{\alpha}_{inf}$ - only negatives values are retrieved from $\boldsymbol{\alpha}$, the rest components are set to zero. Unknowns are cross-sectional areas A_k, $k \in K$ of the elements and vectors of plasticity multipliers $\boldsymbol{\lambda}_{max}$, $\boldsymbol{\lambda}_{min}$.

In case of monotonically increasing load $j = 1$ and conditions (25), (26) of all discretized frame obtain the following form: $\boldsymbol{\varphi} = \boldsymbol{M}_0 - \boldsymbol{\Phi}\left(\boldsymbol{G}\,\boldsymbol{\lambda} + \boldsymbol{S}_e\right) \geq \boldsymbol{0}$, $\boldsymbol{\lambda}^T \boldsymbol{\varphi} = 0$, $\boldsymbol{\lambda} \geq \boldsymbol{0}$. Stiffness constrains (28) of the frame become more simplified: $\boldsymbol{u}_{r,min} \leq \boldsymbol{H}\,\boldsymbol{\lambda} \leq \boldsymbol{u}_{r,max}$. Scope of the problem (25)-(28) becomes reduced and computer realization of the problem is simpler.

It should be noted that numerical solution of the problems (24)-(28), (29)-(34) is easier when complementary slackness conditions are moved to objective function. Then the problem (29)-(34) obtains the following form [16]
find

$$min\left(\sum_k L_k A_k + \boldsymbol{\lambda}_{max}^T \boldsymbol{\varphi}_{max} + \boldsymbol{\lambda}_{min}^T \boldsymbol{\varphi}_{min}\right) \tag{35}$$

subject to

$$\boldsymbol{\varphi}_{max} = \boldsymbol{M}_0 - \boldsymbol{G}\,\boldsymbol{\lambda} - \boldsymbol{M}_{e,max} \geq \boldsymbol{0}$$

$$\boldsymbol{\varphi}_{min} = \boldsymbol{M}_0 + \boldsymbol{G}\,\boldsymbol{\lambda} + \boldsymbol{M}_{e,min} \geq \boldsymbol{0} \tag{36}$$

$$\boldsymbol{\lambda}_{max} \geq \boldsymbol{0}\,,\ \boldsymbol{\lambda}_{min} \geq \boldsymbol{0} \tag{37}$$

$$\boldsymbol{\lambda} = \left(\boldsymbol{\lambda}_{max}, \boldsymbol{\lambda}_{min}\right)^T \tag{38}$$

$$A_k \geq A_{k,min}\,,\quad k \in K \tag{39}$$

$$\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf}\,,\quad \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max} \tag{40}$$

5. Shakedown load optimization of frames

In the case of variable repeated load, the problem of load variation bound $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$ determination is important also. It stated as follows: find shakedown load variation bounds $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$, satisfying the prescribed optimality criterion max $\left\{\boldsymbol{T}_{sup}^T\,\boldsymbol{F}_{sup} + \boldsymbol{T}_{inf}^T\,\boldsymbol{F}_{inf}\right\}$, also strength and stiffness requirements of the structure. Here $\boldsymbol{T}_{sup}$, $\boldsymbol{T}_{inf}$ are the optimality criterion weight coefficient vectors.

Then mathematical model of shakedown load optimization problem for the frames reads
find

$$max\left\{\boldsymbol{T}_{sup}^T\,\boldsymbol{F}_{sup} + \boldsymbol{T}_{inf}^T\,\boldsymbol{F}_{inf} - \sum_j \boldsymbol{\lambda}_j^T \boldsymbol{\varphi}_j\right\} \tag{41}$$

subject to

$$\boldsymbol{\varphi}_j = \boldsymbol{M}_0 - \boldsymbol{\Phi}\left(\boldsymbol{G}\,\boldsymbol{\lambda} + \boldsymbol{S}_{ej}\right) \geq \boldsymbol{0} \tag{42}$$

$$\boldsymbol{\lambda}_j \geq \boldsymbol{0}\,,\ \boldsymbol{\lambda} = \sum_j \boldsymbol{\lambda}_j\,,\ j \in J \tag{43}$$

$$\boldsymbol{F}_{sup} \geq \boldsymbol{0}\,,\ \boldsymbol{F}_{inf} \geq \boldsymbol{0} \tag{44}$$

$$\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf}\,,\ \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max} \tag{45}$$

The vector of limit bending moments $\boldsymbol{M}_0$ and the limits of residual displacements $\boldsymbol{u}_{r,min}$, $\boldsymbol{u}_{r,max}$ are known in the problem (41)-(45). Optimal solution of the problem (41)–(45) is vectors $\boldsymbol{F}^*_{sup}$, $\boldsymbol{F}^*_{inf}$ and $\boldsymbol{\lambda}^*_j$, $j \in J$.

When only bending moments M are taken in to account, the following mathematical model of frame shakedown load optimization is obtained
find

$$max\left\{ \boldsymbol{T}^T_{sup}\,\boldsymbol{F}_{sup} + \boldsymbol{T}^T_{inf}\,\boldsymbol{F}_{inf} - \boldsymbol{\lambda}^T_{max}\,\boldsymbol{\varphi}_{max} - \boldsymbol{\lambda}^T_{min}\,\boldsymbol{\varphi}_{min} \right\} \quad (46)$$

subject to

$$\boldsymbol{\varphi}_{max} = \boldsymbol{M}_0 - \boldsymbol{G}\,\boldsymbol{\lambda} - \boldsymbol{M}_{e,max} \geq \boldsymbol{0}$$

$$\boldsymbol{\varphi}_{min} = \boldsymbol{M}_0 + \boldsymbol{G}\,\boldsymbol{\lambda} + \boldsymbol{M}_{e,min} \geq \boldsymbol{0} \quad (47)$$

$$\boldsymbol{M}_{e,max} = \boldsymbol{\alpha}_{sup}\boldsymbol{F}_{sup} - \boldsymbol{\alpha}_{inf}\boldsymbol{F}_{inf}$$

$$\boldsymbol{M}_{e,min} = -\boldsymbol{\alpha}_{sup}\boldsymbol{F}_{inf} + \boldsymbol{\alpha}_{inf}\boldsymbol{F}_{sup} \quad (48)$$

$$\boldsymbol{F}_{sup} \geq \boldsymbol{0}, \; \boldsymbol{F}_{inf} \geq \boldsymbol{0} \quad (49)$$

$$\boldsymbol{\lambda} = \left(\boldsymbol{\lambda}_{max}, \boldsymbol{\lambda}_{min} \right)^T \quad (50)$$

$$\boldsymbol{\lambda}_{max} \geq \boldsymbol{0}, \; \boldsymbol{\lambda}_{min} \geq \boldsymbol{0} \quad (51)$$

$$\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf}, \; \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max} \quad (52)$$

Load variation bound $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$ and vectors of plasticity multipliers $\boldsymbol{\lambda}_j \geq \boldsymbol{0}$, $j \in J$ are unknowns of nonlinear mathematical programming problem (46)-(52).

6. Numerical examples

6.1. Example 1

The two-storey frame shown in Fig. 5 is subjected by two independent loads: vertical forces of the magnitude $2V$, $3V$ acting in the middle of each beam and horizontal forces $2H$, H. Variation limits of the load are defined by inequalities $0 \leq H \leq H_{sup} = 40\,\text{kN}$, $0 \leq V \leq V_{sup} = 65\,\text{kN}$. The main task is to determine minimal volume of adapted frame (Fig. 5) according to the mathematical models (24)-(28) and (29)-(34), when the frame is made from steel, which elasticity modulus $E = 210$ GPa and the yield limit $\sigma_y = 200$ MPa. Cross-sections of the frame columns and beams are shown in Fig. 6. Parameters b and h' remain the same during all optimization process, only thickness of the flanges is varying. Initial thickness of the flanges is assumed $t^0_{f,col} = 14\,\text{mm}$ for the frame columns and $t^0_{f,beam} = 20\,\text{mm}$ for the beams. Thus, initial cross-sectional areas of the columns and beams are $A^0_{col} = A^0_1 = A^0_2 = 56\,\text{cm}^2$ and $A^0_{beam} = A^0_3 = A^0_4 = 80\,\text{cm}^2$, respectively. Initial volume of the structure is $V^0 = 259200\ \text{cm}^3$. Limit forces of cross-sections are calculated according to the following formulas:

$$M_0 = \sigma_y bth' = \sigma_y A \frac{h'}{2},\; N_0 = \sigma_y 2bt = \sigma_y A$$

Initial limit forces of the columns are $M^0_{0,col} = 160$ kNm and $N^0_{0,col} = 1120$ kN, limit forces of the beams are $M^0_{0,beam} = 320$ kNm and $N^0_{0,beam} = 1600$ kN; relations $c_{col} = 0.2$ and $c_{beam} = 0.125$. Yeld conditions are aproximated by four lines (coefficients of lines described in matrix $\boldsymbol{\Phi}_v$ are shown in Fig. 4).

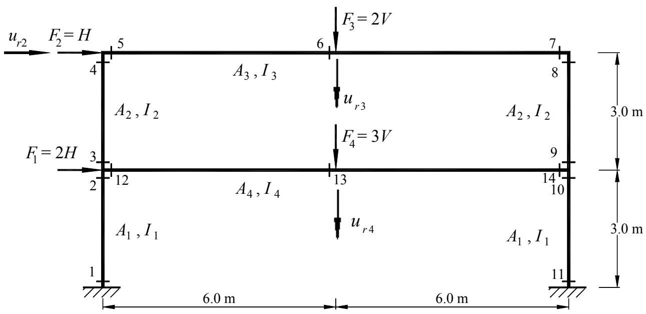


Fig. 5 Discretized frame

Minimal volume searching is performed in the two following cases:

A1 – when the vector of inner forces of discretized frame is $\boldsymbol{S} = \left(\boldsymbol{M}, \boldsymbol{N}\right)^T = \left(M_1, M_2, M_3, ..., M_{14}, N_1, N_2, ..., N_6\right)^T = \left(S_z\right)^T$, $z = 1, 2, ..., n = 20$, i.e. both bending moments M and axial forces N are taken into account.

A2 – when the vector of inner forces of discretized frame is $\boldsymbol{M} = \left(M_z\right)^T = \left(M_1, M_2, M_3, ..., M_{14}\right)^T$, $z = 1, 2, ..., n = 14$, i.e. only bending moments M are evaluated.

In the case A1 frame volume minimization is performed according to the mathematical model (24)-(28). Unknowns are cross-sectional areas of the frame columns and beams A_k, $k \in K$ and vectors of plasticity multipliers $\boldsymbol{\lambda}_j$, $j = 1, 2, 3$. In the case A2 the frame volume minimization problem is solved using the mathematical model (29)-(34). Unknowns are cross-sectional areas A_k, $k \in K$ and vectors of plasticity multipliers $\boldsymbol{\lambda}_{max}$, $\boldsymbol{\lambda}_{min}$.

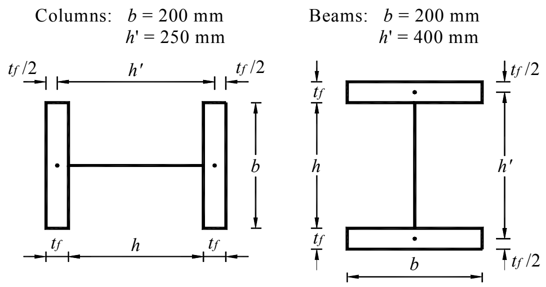


Fig. 6 Geometry of cross-sections

Without any residual displacement constraints (28) or (34), the following minimum volumes of the frame were obtained: $V_{min} = 265288\,\text{cm}^3$ in the case A1 and $V_{min} = 246812\,\text{cm}^3$ in the case A2 (in both cases elastic-plastic state is just before cyclic plastic failure).

Later, the following residual displacement constraints were imposed for displacement $u_{r,2}$ (Fig. 5): $0 \le u_{r,2} \le u_{r,max}$ (here $u_{r,max}$ = 5, 10, 15, 20, 23 mm). Variation of the frame volume depending on prescribed limit on residual displacement $u_{r,max}$ is shown in Fig. 7 for both cases A1 and A2.

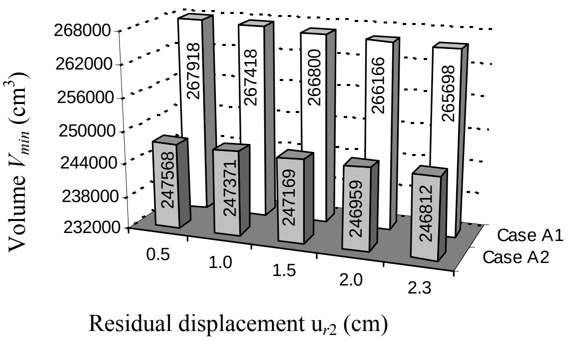


Fig. 7 Variation of minimal volume V_{min} in terms of $u_{r,2}$

6.2 Example 2

The frame is subjected by repeated variable load $0 \le F_2 \le F_{2,sup}$, $0 \le F_3 \le F_{3,sup}$ ir $0 \le F_4 \le F_{4,sup}$. Discretized frame, direction of forces F_2, F_3, F_4 and its application place is shown in Fig. 5. The frame columns HE 300A and beams IPE 450 are made from steel, which elasticity modulus $E = 210$ GPa and yield limit $\sigma_y = 235$ MPa. The main task is to determine maximal load variation bounds $F_{2,sup}$, $F_{3,sup}$ and $F_{4,sup}$, i. e. find max $\left(F_{2,sup} + F_{3,sup} + F_{4,sup}\right)$.

Vector of the inner forces of discretized frame (Fig. 5), when bending moments M and axial forces N are taken into account is: $\boldsymbol{S} = (\boldsymbol{M}, \boldsymbol{N})^T$ $= (M_1, M_2, M_3, ..., M_{14}, \; N_1, N_2, ..., N_6)^T$ $= (S_z)^T$ $z = 1, 2, ..., n = 20$. Limit bending moment M_0 and limit axial force N_0 of the columns and beams are calculated according to the following formulas: $M_0 = \sigma_y W_p$, $N_0 = \sigma_y A$.

Load optimization problem max $\left(F_{2,sup} + F_{3,sup} + F_{4,sup}\right)$ is solved according to the mathematical model (41)–(45), when matrix $\boldsymbol{\Phi}_v$, shown in Fig. 4, is taken into account.

Without residual displacement constraints (45) - i.e. in the state near cyclic plastic failure - the following load variation bounds were obtained: $F^*_{2sup} = 257.47$ kN, $F^*_{3,sup} = 151.56$ kN and $F^*_{4,sup} = 164.65$ kN (max $\left(F_{2,sup} + F_{3,sup} + F_{4,sup}\right) = 573.68$).

When residual displacement constraints (45) $0 \le u_{r,2} \le u_{r2,max} = 10.0$ mm, $0 \le u_{r,3} \le u_{r3,max} = 15.0$ mm and $0 \le u_{r,4} \le u_{r4,max} = 15.0$ mm are evaluated, load variation bounds were obtained: $F_{2sup} = 131.55$ kN, $F_{3,sup} = 189.81$ kN, $F_{4,sup} = 216.49$ kN (max $\left(F_{2,sup} + F_{3,sup} + F_{4,sup}\right) = 537.85$).

7. Conclusions

1. The main difficulty in solving the problem of determinating the optimal parameter distribution of adapted frame is the reasoning of more realistic relation between the area and limit bending moment of different shape cross-sections. For that purpose it is useful to obtain a correlation between the mentioned quantities.

2. There are created mathematical models of the optimization problem for shakedown frames, which evaluate steel plastic deformations and serviceability requirements.

3. There is created a new algorithm that solves problems, which considers for the displacements non-monotonic variation of shakedown frames.

4. There is presented the possibility to use section databases in the real minimal volume frame design problems.

References

1. **Kaneko, L., Maier, G.** Optimum design of plastic structures under displacement's constraints.-Computer Methods in Applied Mechanics and Engineering, 1981, Vol 27 (3), p.369-392.
2. **Stein, E., Zhang, G., Mahnken, R**. Shakedown analysis for perfectly plastic and kinematic hardening materials.-In: CISM. Progress in Computernal Analysis or Inelastic Structures.-Wien, New York: Springer Werlag, 1993, p.175-244.
3. **Giambanco, F., Palizzolo, L., Polizzotto, C.** Optimal shakedown design of beam structures.-Structural Optimization, 1994, v.8, p.156-167.
4. **Tin–Loi, F.** Optimum shakedown design under residual displacement constraints.-Structural and Multidisciplinary Optimization, 2000, v.19(2), p.130-139.
5. **Kaliszky, S., Lógó, J.** Plastic behaviour and stability constraints in the shakedown analysis and optimal design of trusses.-Structural and Multidisciplinary Optimization, 2002, v.24(2), p.118-124.
6. **Choi, SH., Kim, SE.** Optimal design of steel frame using practical nonlinear inelastic analysis.-Engineering Structures, 2002, v.24(9), p.1189-1201.
7. **Staat, M., Heitzer, M (eds).** Numerical methods for limit and shakedown analysis. Series of John von Neumann Institute for Computing, 2003, v.15.-306p.
8. **Benfratello, S., Cirone, L., Giambanco, F.** A multicriterion design of steel frames with shakedown constraints.-Computers and Structures, 2006, v.84, p.269-282.
9. **Cyras, A.A.** Mathematical Models for the Analysis and Optimization of Elastoplastic Structures.-Chichester: Ellis Horwood Lim., 1983.-121p.
10. **Atkočiūnas, J., Borkowski, A., König, JA.** Improved bounds for displacements at shakedown.-Computer Methods in Applied Mechanics and Engineering, 1981, v.28(3), p.365-376.
11. **Dorosz, S., König, JA.** An iterative method of evaluation of elastic-plastic deflections of hyperstatic framed structures.-Ingenieur-Archiv, 1985, v.55, p.202-212.

12. **Maier, G., Comi, C., Corigliano, A., Perego, U., Hübel, H.** Bounds and Estimates on Inelastic Deformations: a Study of their Practical Usefulness. European Commission Report, Nuclear Science and Technology Series, Brussels: European Commission, 1996.-286p.
13. **Hachemi, A., Weichert, D.** Application of shakedown theory to damaging inelastic material under mechanical and thermal loads.-Int. J. of Mechanical Sciences, 1997, v.39(9), p.1067-1076.
14. **Lange–Hansen, P.** Comparative study of upper bound methods for the calculation of residual deformation after shakedown, Series R, No.49.-Lyngby: Technical University of Denmark, Dept. of Structural Engineering and Materials, 1998.-74p.
15. **Merkevičiūtė, D., Atkočiūnas, J.** Optimal shakedown design of metal structures under stiffness and stability constraints.-J.l of Constructional Steel Research, 2006, v./Iss. 62/12, p.1270-1275.
16. **Venskus, A., Atkočiūnas, J.** Improved solution algorithm for shakedown optimization problems.-Material of 9th conference of young Lithuanian scientist "Science - Future of Lithuania", held in Vilnius in March 29-31.-Vilnius, 2006, p.265-270.
17. **Bazaraa, MS., Sherali, HD., Shetty, CM.** Nonlinear programming: theory and algorithms.-New York: Brijbasi Art Press Ltd., John Wiley & Sons, Inc., 2004. -652p.
18. **Atkočiūnas, J., Jarmolajeva, E., Merkevičiūtė, D.** Optimal shakedown loading for circular plates. -Structural and Multidisciplinary Optimization, 2004, v.27(3), p.178-188.
19. **Skaržauskas, V., Merkevičiūtė, D., Atkočiūnas, J.** Optimisation des portiques dans les conditions d'adaptation avec des restrictions en déplacements. -Revue Européenne de Génie Civil, 2005, v.9, No4, p.435-453.
20. **Chraptovič, E., Atkočiūnas, J.** Mathematical programming applications peculiarities in shakedown problem. Civil Engineering, 2001, v.VII, No2, p.106-114.
21. **Ferris, M.C., Tin-Loi, F.** On the solution of a minimum weight elastoplastic problem involving displacement and complementarity constraints.-Comput. Methods Appl. Mech. Engrg. 174, 1999, p.107-120.

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NETIESINIS PROGRAMAVIMAS IR RĖMŲ OPTIMIZACIJA PRISITAIKOMUMO SĄLYGOMIS

R e z i u m ė

Straipsnyje nagrinėjama matematinio programavimo teorija, kuri yra plačiai paplitusi kaip ekstreminių uždavinių sprendimo metodas. Ji talkina prisitaikomumo teorijos optimizavimo uždavinių nagrinėjimui nuo jų matematinių modelių sudarymo iki skaitinio sprendinio rezultatų. Bendrieji optimizavimo uždavinių matematiniai modeliai pritaikyti optimalių idealiai tampriai - plastiškai deformuotų rėmų parametrų arba apkrovos pasiskirstymams prisitaikymo būvyje rasti. Uždaviniai spręsti taikant Rozeno projektuojamųjų gradientų metodą. Pateikta šio metodo optimalumo kriterijaus mechaninė interpretacija. Skaitiniai rėmų optimizacijos rezultatai gauti prisilaikant mažų poslinkių prielaidos.

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NONLINEAR PROGRAMMING AND OPTIMAL SHAKEDOWN DESIGN OF FRAMES

S u m m a r y

This paper considers matematical programming theory, which is widely used as a method of extremum problems solution. It helps for the investigation of shakedown problems from creating of it's mathematical models till receiving numerical solution results. Common mathematical models of optimization are adapted to find optimal parameters or load distribution of elastic perfectly-plastic shakedown frames. Rosen project gradient method is applied to solve the problems. Mechanical interpretation of optimality criterion is presented for the mentioned method. Numerical results of frame optimization problems are received with small displacements assumption.

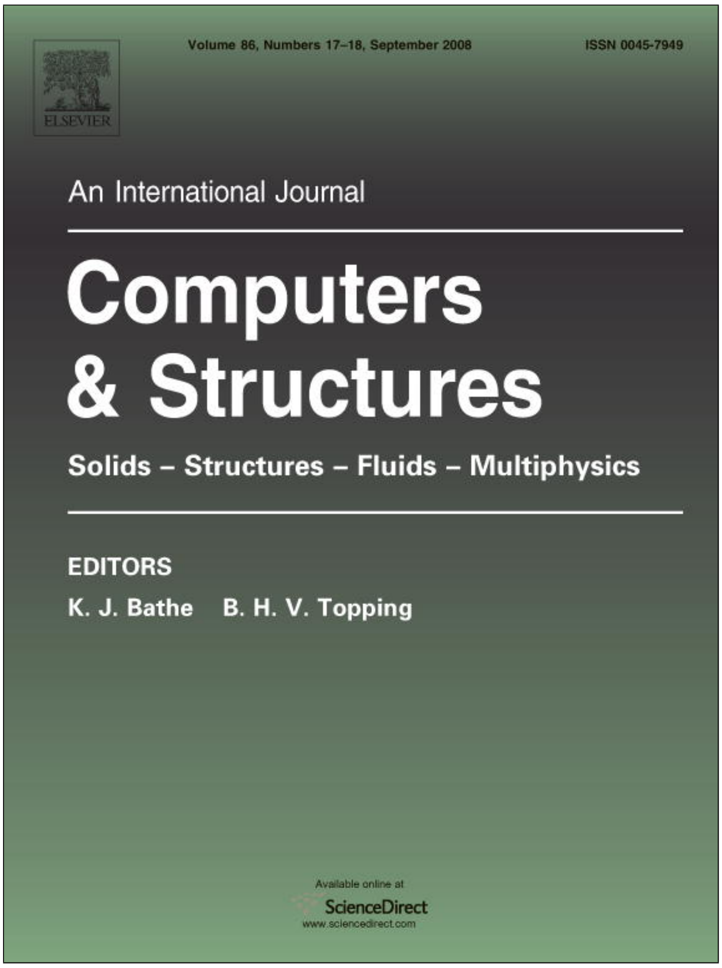
Ю. Аткочюнас, Д. Меркявичюте, А. Венскус,
В. Скаржаускас

НЕЛИНЕЙНОЕ ПРОГРАММИРОВАНИЕ И ОПТИМИЗАЦИЯ РАМ В УСЛОВИЯХ ПРИСЛОСОБЛЯЕМОСТИ

Р е з ю м е

Теория математического программирования, широко распространившаяся как метод решения экстремальных задач, сопутствует исследованию задачи теории пластичности от ее постановки до окончательного решения. В статье общие математические модели оптимизации отнесены к определению оптимального распределения параметров или нагруэки идеально упруго-пластических рам в условиях прислособляемости. Для решения полученых нелинейных задач применен метод проектируемых градиентов Розена. Приведена механическая интерпретация критерев оптимальности этого метода. Численые результаты оптимизации рам получены в рамках теории малых перемещений.

Received December 15, 2006


Volume 86, Numbers 17–18, September 2008
ISSN 0045-7949
ELSEVIER
An International Journal
Computers & Structures
Solids – Structures – Fluids – Multiphysics
EDITORS
K. J. Bathe B. H. V. Topping
Available online at
ScienceDirect
www.sciencedirect.com

Available online at www.sciencedirect.com
ScienceDirect
Computers and Structures 86 (2008) 1757–1768
Computers & Structures
www.elsevier.com/locate/compstruc

Optimal shakedown design of bar systems: Strength, stiffness and stability constraints

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Received 21 May 2007; accepted 24 January 2008
Available online 6 March 2008

Abstract

Using the concept of a variable repeated load and shakedown theory, a unified technique is proposed for formulating mathematical models for the optimization of frame- and truss-like structures under different loads. Strength, stiffness and stability (for trusses only) constraints are included in non-linear mathematical models of structure volume minimization and load optimization problems. Even though the load is prescribed within certain limits, the mathematical models allow the variational bounds of the displacement (the stiffness of the structure depends on them) to be evaluated in the deformed state. Numerical example concerning calculation of frame structure is presented. The results are valid for small displacements.

Keywords: Optimal shakedown design; Elastic–plastic bar systems; Energy principle; Mathematical programming

1. Introduction

This paper, which considers elastic–plastic bar systems (frames, trusses) adapted to a variable repeated load, is an updated and extended version of conference material [1,2]. A variable repeated load is a system of forces that may vary independently within prescribed bounds. Usually variable repeated forces are not characterized by the loading history $\mathbf{F}(t)$, but only by time-independent lower and upper bounds of the forces $\mathbf{F}_{sup}$, $\mathbf{F}_{inf}$, ($\mathbf{F}_{inf} \leqslant \mathbf{F}(t) \leqslant \mathbf{F}_{sup}$).

A variable repeated load and the related concept of shakedown theory not only enable mathematical models for the optimization of elastic–plastic structures at shakedown to be formulated using a unified technique, but also allow these models to be extended to cases of load and effect combinations, and a monotonically increasing or moving load. This possibility of a variable repeated load interpretation is a distinctive feature of this paper.

An adapted structure is safe with respect to cyclic–plastic collapse, but does not satisfy its serviceability requirements, such as those related to stiffness [3–10]. Therefore, not only strength, but also stiffness and even stability requirements ensuring conditional constraints should be included in the mathematical models of the optimal design of structures at shakedown [10]. The stiffness conditions are realized by the restriction of structural deflections or nodal displacements $\mathbf{u} = \mathbf{u}_e + \mathbf{u}_r$ (here the subscripts e and r refer to the elastic and residual parts of the displacement, respectively). The stress–strain state of a dissipative system depends on its loading history. The problem of determining the displacement of an elastic–plastic structure becomes particularly difficult when variable repeated forces $\mathbf{F}(t)$ are prescribed only by their limits of variation $\mathbf{F}_{sup}$, $\mathbf{F}_{inf}$. In this case, it is possible to find only variational bounds $\mathbf{u}_{r,inf}$, $\mathbf{u}_{r,sup}$ of the residual displacements $\mathbf{u}_r(t)$ such that $\mathbf{u}_{r,inf} \leqslant \mathbf{u}_r(t) \leqslant \mathbf{u}_{r,sup}$ [11–20]. Knowing that during the adaptation process the residual displacements $\mathbf{u}_r(t)$ can vary non-monotonically, the determination of the limits of the residual displacement $\mathbf{u}_{r,inf}$, $\mathbf{u}_{r,sup}$ becomes an important constituent of mathematical models of optimization problems. Different references can be found proposing many techniques for calculating the variational bounds of the residual displacement $\mathbf{u}_{r,inf}$, $\mathbf{u}_{r,sup}$ [18]. In this paper,

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0045-7949/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.compstruc.2008.01.008

the technique for determining these bounds is based on compatibility equations of the residual strains and on the solution of a linear programming problem.

Non-linear mathematical models of the volume minimization of an adapted structure and load optimization problems are considered. In the mathematical models of optimization problems, the non-linearity results from the yield conditions (for frames with more complicated cross-sections they are non-linear) and the complementary slackness condition of mathematical programming. The complementary slackness condition does not allow a possible unloading phenomenon of the cross-sections of the structure to be directly fixed. This phenomenon means that after the appearance of plastic strains $\mathbf{\Theta}_{\mathrm{p}}$, the yield condition satisfied as an equality can become an inequality during a future deformation process but the plasticity multiplier remains positive, $\lambda > 0$ [21–23]. The phenomenon of unloading cross-sections leads to a non-monotonic variation of the residual displacement $\mathbf{u}_{\mathrm{r}}(t)$. Only the process of holonomic deformation can be related to the complementary slackness condition of mathematical programming. Unfortunately, the adaptation of a structure is not such a process (it is important to notice that not all the research dealing with shakedown problems pays attention to this). That is why the stiffness conditions, related to the determination of the limits of the residual displacement $\mathbf{u}_{\mathrm{r,inf}}$, $\mathbf{u}_{\mathrm{r,sup}}$ of an adapted structure, should be checked during the solution of the structure optimization problem. Thus, in this paper, the problem of the optimal shakedown design is not a classical one: during the volume minimization of a frame (or truss) it is necessary to determine the variational bounds $\mathbf{u}_{\mathrm{r,inf}}$, $\mathbf{u}_{\mathrm{r,sup}}$ of the residual displacement because of the constant checking of the stiffness conditions. Only in this way is it possible to avoid the influence of the complementary slackness condition of mathematical programming, which does not simulate and in the general case distorts the physical meaning of displacement variation at shakedown.

Using our long experience in the application of the Rosen project gradient method [24–26] for the solution of non-linear optimization problems of elastic–plastic structures, we have developed a new computational procedure for the volume minimization of bar systems at shakedown. This procedure enables structures to be optimized under different load combinations; this is very relevant in civil engineering.

The organization of this paper is as follows. In the next section, the main dependencies of the discretized frame are presented. Section 3 deals with the calculation of the residual forces and displacements of a structure at shakedown (analysis problem). In Section 4, the determination of the variational bounds of the residual displacement is presented in detail. The description of a moving load case is presented in Section 5. Section 6 is devoted to the problem of frame volume minimization at shakedown. Section 7 deals with the optimal shakedown design of trusses. Here the mathematical models are constructed using the ones stated for frames in the earlier sections. Numerical example of minimum volume determination of three-stories frame is presented in Section 8. It shows the peculiarities of the proposed technique. The results were obtained based on the assumption of small displacements.

2. The main dependencies of discretized frames

The geometry of the frame, the cross-sectional shape of the elements and the yield limit of the material σ_{y} are known (it is assumed that the joints of the frame can be fully rigid or fully pinned). The numerical solution of optimization problems is related to the construction of a discrete model of the structure. The frame is discretized by means of s equilibrium finite elements ($k = 1, 2, \ldots, s$, $k \in K$, where K is the set of finite elements), which ensure that the equilibrium equations are exactly satisfied [27–29]. In this case, the approximated forces are the bending moments M and axial forces N. The kth element has s_k nodal points ($l = 1, 2, \ldots, s_k$). The nodal bending moments and axial forces of an element compound an n-vector of generalized forces $\mathbf{S} = (\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_\zeta)^{\mathrm{T}} = (S_z)^{\mathrm{T}}$ and generalized strains compound an n-vector $\mathbf{\Theta} = (\mathbf{\Theta}_1, \mathbf{\Theta}_2, \ldots, \mathbf{\Theta}_\zeta)^{\mathrm{T}} = (\Theta_z)^{\mathrm{T}}$, $\zeta \leqslant s \times s_k$, $z = 1, 2, \ldots, n$. Here ζ is the total number of design sections; in the future, checking of the yield conditions will be performed on these sections. If the degree of freedom of the discretized frame is m, $i = 1, 2, \ldots, m$ (m is the total number of equilibrium equations of the joints and elements) and the vector of forces $\mathbf{S}$ has n components, the order of the coefficient matrix $\mathbf{A}$ of the equilibrium equations $\mathbf{AS} = \mathbf{F}$ is $m \times n$. The number of components of the load vector $\mathbf{F} = (F_1, F_2, \ldots, F_m)^{\mathrm{T}}$ is the same as the degree of freedom of the discretized frame m. It is known from mathematical programming theory that each extreme principle of structural mechanics formulated in terms of forces corresponds to the dual principle expressed in terms of the state of strain. Therefore, in the case of small displacements, it is easy to get equilibrium equations from geometrical equations; then the dual pairs become the forces $\mathbf{S}$ and strains $\mathbf{\Theta}$, and the displacements $\mathbf{u}$ and loads $\mathbf{F}$. That is why the vector of all displacements of the discretized frame $\mathbf{u}$ is variable dual to the load vector $\mathbf{F}$ and is included in the linear geometrical equations $\mathbf{A}^{\mathrm{T}}\mathbf{u} = \mathbf{\Theta}$ (both $\mathbf{F}$ and $\mathbf{u}$ are m-vectors).

The characteristics of the frame's cross-sectional resistance are the limit bending moment $M_0 = \sigma_{\mathrm{y}} W_{\mathrm{pl}}$ and the axial force $N_0 = \sigma_{\mathrm{y}} A$; here W_{pl} is the plastic modulus of a section and A is a cross-sectional area. Though the shapes of cross-sections can be different, the problems in this paper are more oriented towards an I shaped cross-section, i.e. when the shape factor $\mu = 1,15, \ldots, 1,17$ (for a rectangular cross-section $\mu = \frac{W_{\mathrm{pl}}}{W_{\mathrm{e}}} = 1,5$). This allows a more exact approach to elastic perfectly plastic behaviour (Prandtl's diagram). Thus, the following linear yield condition will be used in mathematical models of the problems (Fig. 1):

$$|M| + c|N| \leqslant M_0, \quad c = \frac{M_0}{N_0}. \tag{1}$$

The forces satisfying the equilibrium equations $\mathbf{AS} = \mathbf{F}$ and the yield conditions (1) at each design section $v = 1, 2, \ldots, \zeta$ ($v \in Z$), are called the statically admissible ones.

For shakedown analysis, it is useful to introduce residual forces $\mathbf{S}_r$, displacements $\mathbf{u}_r$ and strains $\mathbf{\Theta}_r$ besides the elastic forces $\mathbf{S}_e$, displacements $\mathbf{u}_e$ and strains $\mathbf{\Theta}_e$:

$$\mathbf{S} = \mathbf{S}_e + \mathbf{S}_r, \quad \mathbf{u} = \mathbf{u}_e + \mathbf{u}_r, \quad \mathbf{\Theta} = \mathbf{\Theta}_e + \mathbf{\Theta}_r. \tag{2}$$

The structure adapts to a variable repeated load if statically admissible time-independent residual forces $\mathbf{S}_r$ resulting from any loading history $\mathbf{F}(t)$, exist [28,29].

Shakedown analysis is based on the assumptions of geometrical linearity (small strains and displacements) and the validity of an associated flow law.

A variable repeated load $\mathbf{F}(t) = (F_1(t), F_2(t), \ldots, F_m(t))^T$ is characterized by its lower and upper limits $\mathbf{F}_{\inf} = (F_{1,\inf}, F_{2,\inf}, \ldots, F_{m,\inf})^T$, $\mathbf{F}_{\sup} = (F_{1,\sup}, F_{2,\sup}, \ldots, F_{m,\sup})^T$, which are not related to the time t. The loading history is unknown, but it fits in the range $\mathbf{F}_{\inf} \leqslant \mathbf{F}(t) \leqslant \mathbf{F}_{\sup}$. The elastic displacements $\mathbf{u}_e(t)$ and forces $\mathbf{S}_e(t)$ of the structure are determined using the influence of the matrixes of displacement and forces, $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}$, respectively:

$$\mathbf{u}_e(t) = \boldsymbol{\beta}\mathbf{F}(t), \quad \mathbf{S}_e(t) = \boldsymbol{\alpha}\mathbf{F}(t), \tag{3}$$

$\boldsymbol{\beta} = (\mathbf{AKA}^T)^{-1}$, $\boldsymbol{\alpha} = \mathbf{K}\ \mathbf{A}^T\boldsymbol{\beta}$, $\mathbf{K} = \mathbf{D}^{-1}$, where $\mathbf{D}$ is a quasi-diagonal flexibility matrix.

When the loading history is unknown, all possible combinations $\mathbf{F}_j$ of the load bounds $\mathbf{F}_{\sup}$, $\mathbf{F}_{\inf}$ should be taken into account for calculating the elastic forces (number of combinations $p = 2^m$):

$$\mathbf{S}_{ej} = \boldsymbol{\alpha}\mathbf{F}_j, \quad \mathbf{F}_{\inf} \leqslant \mathbf{F}_j \leqslant \mathbf{F}_{\sup}, \quad j = 1, 2, \ldots, p(j \in J). \tag{4}$$

In the case of plastic collapse Eq. (4) allow to determine the type of collapse (incremental or alternating plasticity). In the case of two loads F_1, F_2, the domain of the elastic force variation (locus) is shown in Fig. 2. The number of locus apexes is $p = 4$. For each apex $j = 1, 2, 3, 4$ of the locus four inequalities of the yield condition (1) should be written:

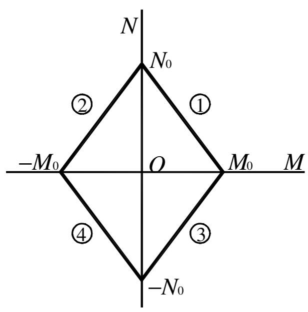


Fig. 1. Linear yield condition.

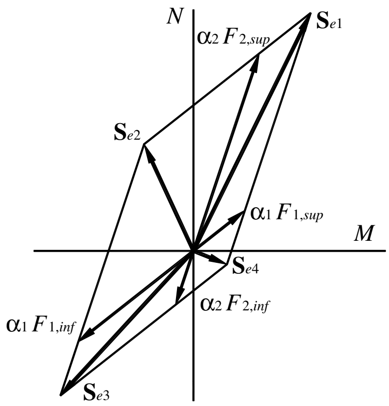


Fig. 2. Domain of elastic force variation.

$$\left.\begin{aligned} f^{(1)}_{kl,j} &= M_{0k} - M_{kl,j} - c_k N_{kl,j} \geqslant 0, \\ f^{(2)}_{kl,j} &= M_{0k} + M_{kl,j} - c_k N_{kl,j} \geqslant 0, \\ f^{(3)}_{kl,j} &= M_{0k} - M_{kl,j} + c_k N_{kl,j} \geqslant 0, \\ f^{(4)}_{kl,j} &= M_{0k} + M_{kl,j} + c_k N_{kl,j} \geqslant 0, \end{aligned}\right\}$$

$$M_{kl,j} = M_{ekl,j} + M_{rkl}, \quad N_{kl,j} = N_{ekl,j} + N_{rkl};$$

$$k = 1, 2, \ldots, s, \quad l = 1, 2, \ldots, s_k, \quad j = 1, 2, \ldots, p. \tag{5}$$

In the expressions (5), it is taken into account that the limit bending moment of an element is $M_{0k} = \text{const}$, $k \in K$; the upper subscript of f is the index of the linear yield condition edge (see Fig. 1). For each design section, the linear yield conditions (5), using matrix $\mathbf{\Phi}_v$, are written as follows:

$$\mathbf{f}_{v,j} = \mathbf{M}_{0v} - \mathbf{\Phi}_v \mathbf{S}_{v,j} \geqslant 0, \quad S_{v,j} = (M_{ev,j} + M_{rv}, N_{ev,j} + N_{rv})^T,$$
$$v = 1, 2, \ldots, \zeta, \quad j = 1, 2, \ldots, p. \tag{6}$$

Here the vector of limit moments $\mathbf{M}_{0v} = (M_{0v}, M_{0v}, M_{0v}, M_{0v})^T$ has the same four components for each section v and the relation $c_k = \frac{M_{0k}}{N_{0k}}$ is prescribed in advance in the 4×2 matrix

$$\mathbf{\Phi}_v = \begin{bmatrix} 1 & c_k \\ -1 & c_k \\ 1 & -c_k \\ -1 & -c_k \end{bmatrix}, \quad v \in V, \quad k \in K.$$

The yield conditions for the whole structure read

$$\mathbf{f}_j = \mathbf{M}_0 - \mathbf{\Phi}\mathbf{S}_j \geqslant 0, \quad \mathbf{S}_j = (\mathbf{M}_{e,j} + \mathbf{M}_r, \mathbf{N}_{e,j} + \mathbf{N}_r)^T, \quad j \in J. \tag{7}$$

Here $\mathbf{f}_j = (\mathbf{f}_{1,j}, \mathbf{f}_{2,j}, \ldots, \mathbf{f}_{\zeta,j})^T$, $\mathbf{\Phi} = \text{diag}\,\mathbf{\Phi}_v$ is a matrix of the linear yield conditions of the whole structure. The vector of limit moment $\mathbf{M}_0 = (\mathbf{M}_{01}, \mathbf{M}_{02}, \ldots, \mathbf{M}_{0\zeta})^T$ is compatible with the yield conditions (6) in dependencies (7).

It is possible directly evaluate not only variable repeated load $\mathbf{F}_j$ but also other loads $\mathbf{F}_c$ (for example self weight of the structure) additionally including them into set J. Elastic forces $\mathbf{S}_{ec}$, resulted by loads $\mathbf{F}_c$, can be included into yield conditions (6) as follows:

$$\mathbf{f}_{v,j} = \mathbf{M}_{0v} - \mathbf{\Phi}_v(\mathbf{S}_{v,j} + \mathbf{S}_{ec}) \geqslant 0, \quad v = 1,2,\ldots,\zeta, \quad j = 1,2,\ldots,p. \tag{8}$$

In the case of $\mathbf{F}_{\inf} = \mathbf{F}_{\sup} = \mathbf{F}$ and $j = 1$ it is possible evaluate only monotonically increasing load.

When the loading history is unknown, vectors of the maximum and minimum values $\mathbf{u}_{e,\sup}$, $\mathbf{u}_{e,\inf}$ of the elastic displacements $\mathbf{u}_e(t) = \boldsymbol{\beta}\mathbf{F}(t)$ are introduced such that $\mathbf{u}_{e,\inf} \leqslant \mathbf{u}_e(t) \leqslant \mathbf{u}_{e,\sup}$. The relation between the displacements $\mathbf{u}_{e,\sup}$, $\mathbf{u}_{e,\inf}$ and load bounds $\mathbf{F}_{\sup}$, $\mathbf{F}_{\inf}$ reads as follows:

$$\mathbf{u}_{e,\sup} = \boldsymbol{\beta}_{\sup}\mathbf{F}_{\sup} + \boldsymbol{\beta}_{\inf}\mathbf{F}_{\inf}, \quad \mathbf{u}_{e,\inf} = \boldsymbol{\beta}_{\sup}\mathbf{F}_{\inf} + \boldsymbol{\beta}_{\inf}\mathbf{F}_{\sup}. \tag{9}$$

Here $\boldsymbol{\beta} = \boldsymbol{\beta}_{\sup} + \boldsymbol{\beta}_{\inf}$, and the components of matrix $\boldsymbol{\beta}_{\sup}$ are positive members of matrix $\boldsymbol{\beta}$ or equal to zero.

The residual forces $\mathbf{S}_r^* = (\mathbf{M}_r^*, \mathbf{N}_r^*)^T$ and displacements $\mathbf{u}_r^*$ of the shakedown state are obtained via the solution of the stress–strain analysis problem [6,14,22].

3. Analysis of the residual force and displacement at shakedown

The residual force and displacement of an adapted frame can be analysed when the load variation bounds $\mathbf{F}_{\sup}$, $\mathbf{F}_{\inf}$ (i.e. elastic forces $\mathbf{S}_{ej}$, $j \in J$), limit moments $\mathbf{M}_0$ and the relation $c_k = \frac{M_{0k}}{N_{0k}}$ ($k \in K$) are given. The residual forces $\mathbf{S}_r = (\mathbf{M}_r, \mathbf{N}_r)^T$ and displacements $\mathbf{u}_r$ of the adapted frame are to be found when it adapts to a variable repeated load $\mathbf{F}_{\inf} \leqslant \mathbf{F}(t) \leqslant \mathbf{F}_{\sup}$. The mathematical model of the analysis problem is formulated on the basis of the minimum complementary deformation energy principle [5,6,20]:

find

$$\min \quad \mathsf{F}'(\mathbf{S}_r) = \min \frac{1}{2}\mathbf{S}_r^T\mathbf{D}\mathbf{S}_r \tag{10}$$

$$\text{subject to} \quad \mathbf{A}\mathbf{S}_r = 0, \tag{11}$$

$$f_j = \mathbf{M}_0 - \mathbf{\Phi}\mathbf{S}_j \geqslant 0, \quad \mathbf{S}_j = \mathbf{S}_{ej} + \mathbf{S}_r \quad \text{for all } j \in J. \tag{12}$$

F' is the objective function of the problem (10)–(12). As mentioned above, the blocks of the quasi-diagonal matrix $\mathbf{\Phi}$ are matrixes of the section yield conditions $\mathbf{\Phi}_v$, $v \in Z$. The optimal solution $\mathbf{S}_r^*$ of the quadratic programming problem (10)–(12) is unique. Though a particular loading history is not considered, an $\mathbf{F}(t)$ in the range $\mathbf{F}_{\inf} \leqslant \mathbf{F}(t) \leqslant \mathbf{F}_{\sup}$ exists that ensures the shakedown state after the appearance of residual forces $\mathbf{S}_r^*$.

The dual problem to the initial one (10)–(12) is stated as follows:

find

$$\max \quad \mathsf{F}''(\mathbf{S}_r, \mathbf{u}_r, \boldsymbol{\lambda}_j) = \max\left\{-\frac{1}{2}\mathbf{S}_r^T\mathbf{D}\mathbf{S}_r - \sum_{j=1}^{p}\boldsymbol{\lambda}_j^T\mathbf{\Phi}\mathbf{S}_r - \sum_{j=1}^{p}\boldsymbol{\lambda}_j^T(\mathbf{M}_0 - \mathbf{\Phi}(\mathbf{S}_{ej} + \mathbf{S}_r))\right\} = \max\left\{-\frac{1}{2}\mathbf{S}_r^T\mathbf{D}\mathbf{S}_r - \sum_{j=1}^{p}\boldsymbol{\lambda}_j^T(\mathbf{M}_0 - \mathbf{\Phi}\mathbf{S}_{ej})\right\} \tag{13}$$

$$\text{subject to} \quad \mathbf{D}\mathbf{S}_r + \sum_{j=1}^{p}\mathbf{\Phi}^T\boldsymbol{\lambda}_j - \mathbf{A}^T\mathbf{u}_r = 0, \tag{14}$$

$$\boldsymbol{\lambda}_j \geqslant 0, \quad j \in J. \tag{15}$$

Here F'' is the objective function of the problem (13)–(15), the dependencies (14) are the geometrical equations $\mathbf{\Theta}_r - \mathbf{A}^T\mathbf{u}_r = 0$, and $\mathbf{\Theta}_p = \sum_{j=1}^{p}\mathbf{\Phi}^T\boldsymbol{\lambda}_j = \mathbf{\Phi}^T\sum_{j=1}^{p}\boldsymbol{\lambda}_j$ are the plastic strains. The optimal solution of the kinematic analysis problem formulation (13)–(15) is $\mathbf{S}_r^*$, $\mathbf{u}_r^*$, $\boldsymbol{\lambda}_j^*$ and also $\mathbf{\Theta}_p^* = \mathbf{\Phi}^T\sum_{j=1}^{p}\boldsymbol{\lambda}_j^*$, $j \in J$. The maximum value of the energy dissipated during the shakedown process is $D_{\max} = \sum_{j=1}^{p}\boldsymbol{\lambda}_j^{*T}\mathbf{M}_0$, $j \in J$.

However, the deformed state of the adapted structure depends on its loading history, i.e. on time t. In other words, the vector of plastic strains $\overline{\mathbf{\Theta}}_p^*$ may be non-unique, resulting in the same residual forces $\mathbf{S}_r^*$ but different residual displacements $\bar{\mathbf{u}}_r^*$. Reselecting the components of all the obtained vectors $\bar{\mathbf{u}}_r^*$, the vectors of the minimum and maximum values $\bar{\mathbf{u}}_{r,\inf}^*$, $\bar{\mathbf{u}}_{r,\sup}^*$ are constructed. Unfortunately, the mathematical model (13)–(15) does not simulate the possibility of finding all the vectors of plastic strains $\overline{\mathbf{\Theta}}_p^*$ here with vectors $\bar{\mathbf{u}}_r^*$. Thus, the main reason for solving the problem (13)–(15) is to determine the magnitude of the energy dissipation $D_{\max}$ (which is widely explained in Section 4).

An adapted frame is safe with respect to cyclic–plastic collapse (alternating plasticity or incremental collapse). It is important to mention that the shakedown of a structure is not determined by the minimum (maximum) value ($\min \mathsf{F}'(\mathbf{S}_r^*) = \max \mathsf{F}''(\mathbf{S}_r^*, \mathbf{u}_r^*, \boldsymbol{\lambda}_j^*)$) of functions (10), (13) but by the fact that any statically admissible forces $\mathbf{S}_r$ (satisfying relations (11) and (12)) of any kinematically admissible displacements $\mathbf{u}_r$ (satisfying relations (14) and (15)) exist [30]. In terms of mathematical programming, this means that the structure will shakedown if the set of admissible solutions of the problems (10)–(14) is not empty [22,24].

In Section 2, it was shown that the pseudo-elastic state of a structure is defined by the vectors $\mathbf{S}_{ej}$, $\mathbf{u}_{e,\inf}$, $\mathbf{u}_{e,\sup}$. When the load bounds $\mathbf{F}_{\inf}$, $\mathbf{F}_{\sup}$ are given, these vectors can be found in advance according to formulas (4), (9), independently of the shakedown analysis. Meanwhile the residual forces $\mathbf{S}_r$, strains $\mathbf{\Theta}_r$ and displacement $\mathbf{u}_r$ satisfy the equations

$$\mathbf{A}\mathbf{S}_r = 0, \quad \mathbf{A}^T\mathbf{u}_r = \mathbf{\Theta}_r, \quad \mathbf{\Theta}_r = \mathbf{D}\mathbf{S}_r + \mathbf{\Theta}_p. \tag{16}$$

Having solved Eq. (16), the expressions of residual forces $\mathbf{S}_r$ and displacements $\mathbf{u}_r$ are obtained in terms of the plastic strains $\mathbf{\Theta}_p$: $\mathbf{S}_r = \overline{\mathbf{G}}\mathbf{\Theta}_p$, $\mathbf{u}_r = \overline{\mathbf{H}}\mathbf{\Theta}_p$. The influence matrixes $\overline{\mathbf{G}}$, $\overline{\mathbf{H}}$ of the residual forces $\mathbf{S}_r$ and residual displacements $\mathbf{u}_r$ read

$$\overline{\mathbf{G}} = \boldsymbol{\alpha}\mathbf{A}\mathbf{K} - \mathbf{K}, \quad \overline{\mathbf{H}} = \boldsymbol{\alpha}^T. \tag{17}$$

$\mathbf{\Theta}_p$ are plastic strains in the formulas for calculating the force $\mathbf{S}_r$ and displacement $\mathbf{u}_r$. If the plastic strains $\mathbf{\Theta}_p^* = \mathbf{\Phi}^T\sum_{j=1}^p \boldsymbol{\lambda}_j^*$ that appeared during the deformation process are known, then the residual forces $\mathbf{S}_r^*$ and displacements $\mathbf{u}_r^*$ can be calculated according to the following formulas: $\mathbf{S}_r^* = \overline{\mathbf{G}}\mathbf{\Theta}_p^* = \overline{\mathbf{G}}\mathbf{\Phi}^T\sum_{j=1}^p \boldsymbol{\lambda}_j^* = \mathbf{G}\sum_{j=1}^p \boldsymbol{\lambda}_j^* = \mathbf{G}\boldsymbol{\lambda}^*$, $\mathbf{u}_r^* = \overline{\mathbf{H}}\mathbf{\Theta}_p^* = \mathbf{H}\boldsymbol{\lambda}^*$, $\boldsymbol{\lambda}^* = \sum_{j=1}^p \boldsymbol{\lambda}_j^*$, $j \in J$ [31]. It remains to mention that the influence matrixes $\mathbf{G}$ and $\mathbf{H}$ depend not only on the geometry and physical parameters of the structure but also on the approximation matrix of the yield surface $\mathbf{\Phi}$.

The main difference between elastic–plastic structures subjected to a monotonically increasing loading $\mathbf{F}$ and a variable repeated one $\mathbf{F}(t)$ ($\mathbf{F}_{inf} \leqslant \mathbf{F}(t) \leqslant \mathbf{F}_{sup}$) is the possible appearance of the unloading phenomenon in the sections of the adapting structure. More details about the unloading phenomenon will follow. Plastic strains $\mathbf{\Theta}_{pv}$ occur in section v when the complementary slackness conditions of mathematical programming are satisfied:

$$\boldsymbol{\lambda}_{v,j}^T(\mathbf{M}_{0v} - \mathbf{\Phi}_v\mathbf{S}_{j,v}) = 0 \ (\text{or } \boldsymbol{\lambda}_{v,j}^T f_{v,j} = 0), \quad \boldsymbol{\lambda}_{v,j} \geqslant 0, \quad v \in Z, j \in J. \tag{18}$$

The yield condition satisfied as an equality $f = 0$ can become an inequality $f < 0$ during a future deformation process, but the plasticity multiplier remains positive, $\lambda > 0$. Such behaviour of the structure cannot be evaluated because of the complementary slackness conditions $\boldsymbol{\lambda}_{v,j}^T\mathbf{f}_{v,j} = 0$, $v \in Z$, $j \in J$ (these conditions are included in the objective function (13) of the problem's kinematic formulation (13)–(15)). This is important, because during the adaptation process the residual displacements $\mathbf{u}_r(t)$ can vary non-monotonically – they may increase then later decrease etc. To evaluate the non-monotonic variation of the residual displacements, vectors of the minimum and maximum values $\mathbf{u}_{r,inf}$, $\mathbf{u}_{r,sup}$ are introduced; they are not related to the time t. Vectors of the displacement bounds $\bar{\mathbf{u}}_{r,inf}^*$, $\bar{\mathbf{u}}_{r,sup}^*$ are obtained analysing all possible loading histories $\mathbf{F}(t)$. Meanwhile vectors $\mathbf{u}_{r,inf}$, $\mathbf{u}_{r,sup}$ are approximate, safe bounds of the residual displacement such that

$$\mathbf{u}_{r,inf} \leqslant \bar{\mathbf{u}}_{r,inf}^*, \quad \bar{\mathbf{u}}_{r,sup}^* \leqslant \mathbf{u}_{r,sup}. \tag{19}$$

The stiffness conditions (restriction of displacements of deflections) read:

$$\mathbf{u}_{min} \leqslant \mathbf{u} \leqslant \mathbf{u}_{max}. \tag{20}$$

The vectors $\mathbf{u}_{min}$, $\mathbf{u}_{max}$, used in formula (20), are admissible bounds of the displacement variation defined by construction regulations; they are always known in advance. The stiffness conditions (20) can obtained in the following form:

$$\mathbf{u}_{min} \leqslant \mathbf{u}_e(t)_+\mathbf{u}_r(t) \leqslant \mathbf{u}_{max}.$$

If a particular loading history is not considered, for instance, by incremental methods [21,23], these constraints due to expressions (9) can be rewritten as follows:

$$\mathbf{u}_{min} \leqslant \mathbf{u}_{e,inf} + \mathbf{u}_{r,inf}, \quad \mathbf{u}_{e,sup} + \mathbf{u}_{r,sup} \leqslant \mathbf{u}_{max}. \tag{21}$$

Often only the residual displacements are restricted in volume minimization problems, as the elastic components $\mathbf{u}_{e,inf}$, $\mathbf{u}_{e,sup}$ can be easily calculated according the formulas (9). Then the stiffness conditions (21) read:

$$\mathbf{u}_{r,min} \leqslant \mathbf{u}_{r,inf}, \quad \mathbf{u}_{r,sup} \leqslant \mathbf{u}_{r,max}. \tag{22}$$

The optimal solution $\mathbf{S}_r^*$, D_{max} of the analysis problem (13)–(15) helps to formulate a mathematical model for determining the bounds $\mathbf{u}_{r,inf}$, $\mathbf{u}_{r,sup}$ and to obtain the numerical values of these bounds.

4. Problems of determining the variational bounds of the residual displacement

4.1. The first problem

The components $\tilde{u}_{ri,inf}, \tilde{u}_{ri,sup}$, $i = 1, 2, \ldots, m$ of the variational bound vectors $\tilde{\mathbf{u}}_{r,inf}$, $\tilde{\mathbf{u}}_{r,sup}$ of the residual displacement are obtained by solving the following linear programming problem:

find

$$\begin{matrix}\max \\ \min\end{matrix} \quad \mathbf{H}_i^*\tilde{\boldsymbol{\lambda}} = \begin{bmatrix}\tilde{u}_{ri,sup} \\ \tilde{u}_{ri,inf}\end{bmatrix}, \quad i = 1, 2, \ldots, m, \tag{23}$$

$$\text{subject to} \quad \mathbf{B}_\lambda^*\tilde{\boldsymbol{\lambda}} = \mathbf{B}_r\mathbf{S}_r^*, \quad \tilde{\boldsymbol{\lambda}} \geqslant 0, \tag{24}$$

$$\tilde{\boldsymbol{\lambda}}^T\widetilde{\mathbf{M}}_0 \leqslant D_{max}. \tag{25}$$

This mathematical model represents a fictitious structure, i.e. a system having displacements $\tilde{\mathbf{u}}_{r,inf}$, $\tilde{\mathbf{u}}_{r,sup}$, which "envelope" the displacements $\mathbf{u}_r$ of the given structure at shakedown [22,32] and conjoin main dependencies of the static (10)–(12) and the kinematic (13)–(15) formulations of analysis problem. The unknown of the problem (23)–(25) is ζ-vector $\tilde{\boldsymbol{\lambda}} \geqslant 0$, while the vectors $\mathbf{S}_r^*$, $\widetilde{\mathbf{M}}_0$ and D_{max} are known. Vector $\mathbf{S}_r^*$ and the magnitude of D_{max} are obtained according to the optimal solutions of the problem (13)–(15). $\widetilde{\mathbf{M}}_0$ is a vector of the limit moments of the fictitious structure. The components of vector $\widetilde{\mathbf{M}}_0$ are such that at least one yield condition would be satisfied as a strict equality in each section $v \in Z$ of the frame. Thus, the limit moment of the structure section $\widetilde{\mathbf{M}}_{0v}$ is calculated according to

$$\widetilde{M}_{0v} = \max \mathbf{\Phi}_v(\mathbf{S}_r^* + \mathbf{S}_{ev,j}) \geqslant 0, \quad v \in Z, \quad j \in J. \tag{26}$$

The elastic forces $\mathbf{S}_e^*$ and matrix $\mathbf{\Phi}^*$ of such linear yield conditions $\mathbf{f}_j = \mathbf{M}_0 - \mathbf{\Phi}\mathbf{S}_j \geqslant 0$, which satisfy condition (26), are determined together with the vector $\widetilde{\mathbf{M}}_0$. Then the following equality is valid:

$$\widetilde{\mathbf{M}}_0 = \mathbf{\Phi}^*(\mathbf{S}_r^* + \mathbf{S}_e^*). \tag{27}$$

Thus, in formula (27), the number of vector $\widetilde{\mathbf{M}}_0 = (\widetilde{M}_{01}, \widetilde{M}_{02}, \ldots, \widetilde{M}_{0v}, \ldots, \widetilde{M}_{0\zeta})^T$ components and rows of matrix

$\mathbf{\Phi}^*$ is equal to the number of design sections ζ ($\zeta \leqslant s \times s_k$). The main purpose of applying formula (26) is to construct a new matrix of the yield conditions $\mathbf{\Phi}^*$, which has ζrows and n columns. The matrix $\mathbf{\Phi}^*$ is used for formulating the objective function (23) and condition (24) of the problem (23)–(25). The matrix $\mathbf{H}^*$ used in the objective function is calculated according to the formula $\mathbf{H}^* = \overline{\mathbf{H}}\mathbf{\Phi}^{*\mathrm{T}}$. Equalities (24), $\mathbf{B}^*_\lambda \tilde{\boldsymbol{\lambda}} = \mathbf{B}_\mathrm{r}\mathbf{S}^*_\mathrm{r}$, are compatibility equations of the structure's residual strains. The number of equations is equal to the degree of static indeterminacy of the system $k_0 = n - m$. The compatibility equations of the strains $\mathbf{B}\,\mathbf{\Theta}_\mathrm{p} = \mathbf{B}_\mathrm{r}\mathbf{S}_\mathrm{r}$ are obtained from the geometrical equations $\mathbf{A}^\mathrm{T}\mathbf{u}_\mathrm{r} = \mathbf{D}\mathbf{S}_\mathrm{r} + \mathbf{\Theta}_\mathrm{p}$ after the elimination of displacements $\mathbf{u}_\mathrm{r}$. Here matrixes $\mathbf{B}$ and $\mathbf{B}_\mathrm{r}$ are $\mathbf{B} = [\mathbf{A}^{\mathrm{T}''}(\mathbf{A}^{\mathrm{T}'})^{-1}, -\mathbf{I}]$, $\mathbf{B}_\mathrm{r} = -\mathbf{A}^{\mathrm{T}''}(\mathbf{A}^{\mathrm{T}'})^{-1}\mathbf{D}' + \mathbf{D}''$. Matrixes $\mathbf{A}^{\mathrm{T}'}$, $\mathbf{A}^{\mathrm{T}''}$ and $\mathbf{D}'$, $\mathbf{D}''$ are sub-matrixes of $\mathbf{A}^\mathrm{T}$ and $\mathbf{D}$, respectively; $\mathbf{I}$ is the identity matrix. Using matrix $\mathbf{\Phi}^*$, the equalities $\mathbf{B}^*_\lambda \tilde{\boldsymbol{\lambda}} = \mathbf{B}_\mathrm{r}\mathbf{S}^*_\mathrm{r}$ are obtained, where $\mathbf{B}^*_\lambda = \mathbf{B}\mathbf{\Phi}^{*\mathrm{T}}$.

The vector components of problem (23)–(25) with the optimal solution $\tilde{\boldsymbol{\lambda}}^* \geqslant 0$ are not related to the fulfilment of the complimentary slackness conditions (18) and they may not have the physical meaning of plasticity multipliers (in contrast to the solution $\boldsymbol{\lambda}^* \geqslant 0$ of the problem (13)–(15)). The upper bound of the dissipated energy $D_{\max}$ can also be calculated by Koiter's suggested formula [33]. The fictitious structure method allows a more exact determination of the residual displacement variational bounds $\tilde{\mathbf{u}}_{\mathrm{r,inf}}$, $\tilde{\mathbf{u}}_{\mathrm{r,sup}}$ compared with Koiter's global conditions.

4.2. The second problem

The values $u_{\mathrm{r}i,\mathrm{inf}}$, $u_{\mathrm{r}i,\mathrm{sup}}$ $i = 1, 2, \ldots, m$ of the displacement limits $\mathbf{u}_{\mathrm{r,inf}}$, $\mathbf{u}_{\mathrm{r,sup}}$ can be obtained from the basic solution vectors of $\boldsymbol{\lambda}_0 \geqslant \mathbf{0}$ of the strain compatibility equations $\mathbf{B}^*_\lambda \boldsymbol{\lambda}_0 = \mathbf{B}_\mathrm{r}\mathbf{S}^*_\mathrm{r}$. The basic variables $\lambda'_0 \geqslant 0$ compounding the vector $\boldsymbol{\lambda}_0 \geqslant 0$ can be determined according to $\lambda'_0 = (\mathbf{B}'^*_\lambda)^{-1}\mathbf{B}_\mathrm{r}\mathbf{S}^*_\mathrm{r}$. Here the quadratic $k_0 \times k_0$ matrix $\mathbf{B}'^*_\lambda$ is a sub-matrix of $\mathbf{B}^*_\lambda$. If the determinant of matrix $\mathbf{B}'^*_\lambda$ is equal to zero, the statically determinate system corresponding to $\mathbf{B}'^*_\lambda$ is geometrically unstable. In the general case, the number η of combinations constructing the sub-matrixes $\mathbf{B}'^*_\lambda$ can be smaller or equal to $\zeta!/[k_0!(\zeta - k_0)!]$. After all η vectors $\boldsymbol{\lambda}_0 \geqslant 0$ (here subscript η is omitted) are found, only those vectors satisfying energy condition (25) are selected. If $\boldsymbol{\lambda}_{0,z} \geqslant 0$ satisfies conditions (25), the set of subscripts z is Ξ. The vectors of residual displacements $\mathbf{u}_{\mathrm{r}0,z}$ are calculated according to

$$\mathbf{u}_{\mathrm{r}0,z} = \mathbf{H}^*\boldsymbol{\lambda}_{0z}, \quad z \in \Xi. \tag{28}$$

The vectors $\mathbf{u}_{\mathrm{r,inf}}$, $\mathbf{u}_{\mathrm{r,sup}}$ are constructed by picking the components of all vectors $\mathbf{u}_{\mathrm{r}0,z}$ ($z \in \Xi$) with maximal and minimal values. It is easy to see that one of the vectors $\boldsymbol{\lambda}_{0,z} \geqslant 0$ will coincide with the optimal solution $\boldsymbol{\lambda}^* \geqslant 0$ of the problem (13)–(15), i.e. $\boldsymbol{\lambda}_{0,z} = \boldsymbol{\lambda}^*$. Thus it is possible to write a group of inequalities:

$$\tilde{\mathbf{u}}_{\mathrm{r,inf}} \leqslant \mathbf{u}_{\mathrm{r,inf}} \leqslant \mathbf{u}_\mathrm{r}(t) \leqslant \mathbf{u}_{\mathrm{r,sup}} \leqslant \tilde{\mathbf{u}}_{\mathrm{r,sup}}. \tag{29}$$

Taking into account inequalities (19), the following sequence of inequalities is obtained:

$$\tilde{\mathbf{u}}_{\mathrm{r,inf}} \leqslant \mathbf{u}_{\mathrm{r,inf}} \leqslant \bar{\mathbf{u}}^*_{\mathrm{r,inf}} \leqslant \mathbf{u}_\mathrm{r}(t) \leqslant \bar{\mathbf{u}}^*_{\mathrm{r,inf}} \leqslant \mathbf{u}_{\mathrm{r,sup}} \leqslant \tilde{\mathbf{u}}_{\mathrm{r,sup}}. \tag{30}$$

The compatibility equations of residual strains (24) included in the problem (23)–(25) as constraints can be derived using the formulas $\overline{\mathbf{G}}\mathbf{\Theta}_\mathrm{p} = \mathbf{S}_\mathrm{r}$, $\mathbf{\Theta}_\mathrm{p} = \mathbf{\Phi}^{*\mathrm{T}}\tilde{\boldsymbol{\lambda}}$ and matrix $\mathbf{B}_\mathrm{r}$ as follows:

$$\overline{\mathbf{G}}\mathbf{\Phi}^{*\mathrm{T}}\tilde{\boldsymbol{\lambda}} = \mathbf{S}^*_\mathrm{r}, \tag{31}$$

$$\mathbf{B}_\mathrm{r}\overline{\mathbf{G}}\mathbf{\Phi}^{*\mathrm{T}}\tilde{\boldsymbol{\lambda}} = \mathbf{B}_\mathrm{r}\mathbf{S}^*_\mathrm{r}, \tag{32}$$

and the compatibility equations of the residual strains $\mathbf{B}^*_\lambda\tilde{\boldsymbol{\lambda}} = \mathbf{B}_\mathrm{r}\mathbf{S}^*_\mathrm{r}$ are obtained, where matrix $\mathbf{B}^*_\lambda = \mathbf{B}_\mathrm{r}\overline{\mathbf{G}}\mathbf{\Phi}^{*\mathrm{T}}$.

It is possible to change the constraints (24) of the problem of optimizing the variational bounds of the residual displacement (23)–(25) into condition (31) $\overline{\mathbf{G}}\mathbf{\Phi}^{*\mathrm{T}}\tilde{\boldsymbol{\lambda}} = \mathbf{S}^*_\mathrm{r}$, $\tilde{\boldsymbol{\lambda}} \geqslant 0$, having eliminated the linearly dependant equations in advance. However, it is more practical to use the compatibility equations of residual strains (24): the physical meaning of the second problem of determining the residual displacement variational bounds $\mathbf{u}_{\mathrm{r,inf}}$, $\mathbf{u}_{\mathrm{r,sup}}$ becomes evident.

Both vectors $\mathbf{u}_{\mathrm{r,inf}}$, $\mathbf{u}_{\mathrm{r,sup}}$ and $\tilde{\mathbf{u}}_{\mathrm{r,inf}}$, $\tilde{\mathbf{u}}_{\mathrm{r,sup}}$ can be used in the stiffness constraints (30) of mathematical models of optimization problems.

5. Case of a moving load

A monotonically increasing load is described in this way: $\mathbf{F} = \mathbf{F}_{\mathrm{inf}} = \mathbf{F}_{\mathrm{sup}}$, i.e. the lower and upper bounds coincide. In this instance, the number of elastic force locus apexes is equal to one and the elastic forces are $\mathbf{S}_\mathrm{e}$ ($j = 1$, this index is omitted). If, for example, $\mathbf{F}_{\mathrm{inf}} = 0$, and the components of vector $\mathbf{F}_{\mathrm{sup}}$ take in series the same values, then we get vectors $\mathbf{F}_\xi$ that correspond to each position ξ of the moving force system. In Fig. 3 a system of two forces (F_1 and F_2) moving on the bottom bars of a truss and a load vector $\mathbf{F}_\xi$ corresponding to each position ξ ($\xi = 1, 2, \ldots, \bar{p}$) is shown. For the sake of simplicity the components of $\mathbf{F}_\xi = (F_{1\xi}, F_{2\xi}, \ldots, F_{4\xi})^\mathrm{T}$ are related not to the degree of freedom m of the discretized truss model, but only to the vertical forces of the bottom bars of the truss. The elastic forces of locus apexes $\mathbf{S}_{\mathrm{e}\xi}$ of the construction in the case of a moving load are calculated by formula (4), replacing the index ξ by j and thus considering $p = \bar{p}$ [34].

6. Mathematical models of adapted frame optimization

6.1. Design of minimum-volume frame at shakedown

A minimum-volume frame is designed when the yield limit σ_{yk} of the frame material and the lengths L_k of all its elements k ($k \in K$) and load variation bounds $\mathbf{F}_{\mathrm{sup}}$, $\mathbf{F}_{\mathrm{inf}}$ are known. The problem of frame volume minimization reads: minimize $\sum_k L_k A_k$, subject to structure strength and stiffness constraints. As stated above, the relation $c_k = \frac{M_{0k}}{N_{0k}}$, $k \in K$ should be prescribed in advance. The limit moment of element $M_{0k} = \sigma_{yk} W_{\mathrm{pl},k} = \xi(\sigma_{yk}, A_k)$ is a func-

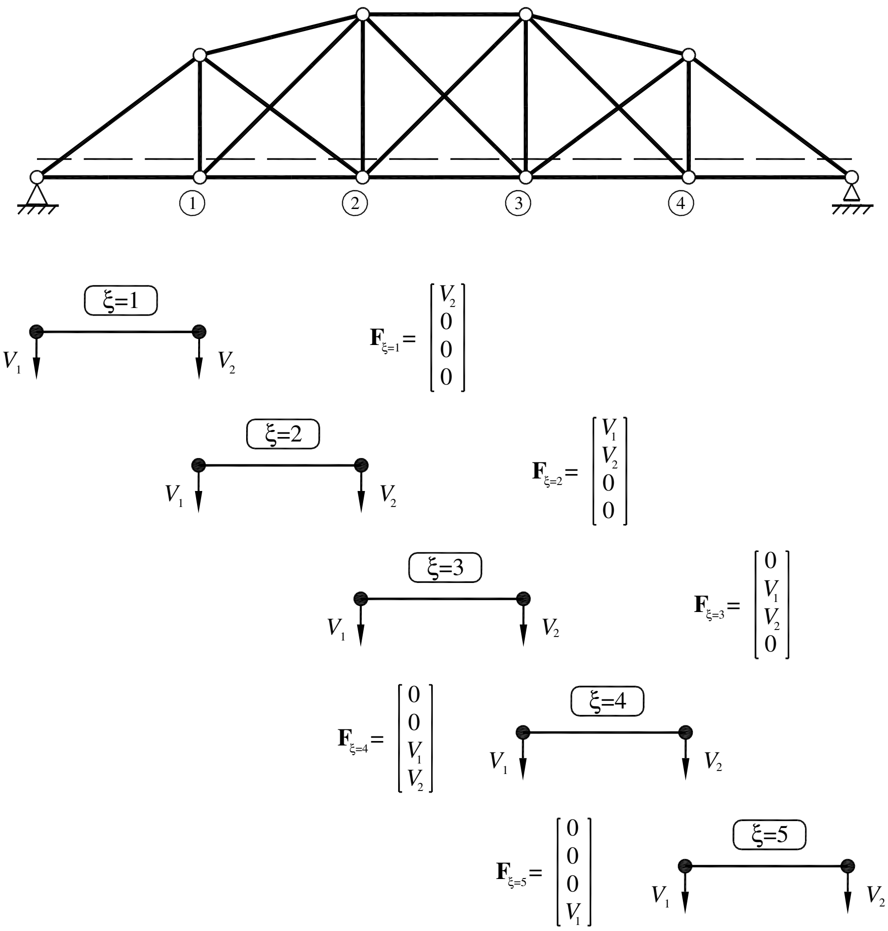


Fig. 3. Moving load realized by vectors $\mathbf{F}_\xi$ ($\xi = 1, 2, \ldots, 5$).

tion of the cross-sectional area A_k and the yield limit of the material σ_{yk}. It is true that usually one or other specific dimension of the cross-section (for instance, the flange thickness t_f of the I-section while the width of flange b is fixed; see Section 8) participates in the functional relation $M_{0k} = \xi(\sigma_{yk}, A_k)$ instead of the cross-sectional area A_k. Then the mathematical model of minimizing the frame volume is as follows:

find

$$\min \quad \sum_k L_k A_k, \tag{33}$$

$$\text{subject to} \quad \mathbf{f}_j = \mathbf{M}_0 - \mathbf{\Phi}(\mathbf{G}\boldsymbol{\lambda} + \mathbf{S}_{ej}) \geqslant 0, \tag{34}$$

$$\sum_{j=1}^{p} \boldsymbol{\lambda}_j^{\mathrm{T}}[\mathbf{M}_0 - \mathbf{\Phi}(\mathbf{G}\boldsymbol{\lambda} + \mathbf{S}_{ej})] = 0, \quad \boldsymbol{\lambda}_j \geqslant 0,$$

$$\boldsymbol{\lambda} = \sum_{j=1}^{p} \boldsymbol{\lambda}_j, \quad j \in J, \tag{35}$$

$$A_k \geqslant A_{k,\min}, \quad k \in K, \tag{36}$$

$$\mathbf{u}_{r,\min} \leqslant \mathbf{u}_{r,\inf}, \quad \mathbf{u}_{r,\sup} \leqslant \mathbf{u}_{r,\max} \tag{37}$$

The cross-sectional areas A_k, $k \in K$ (or another specific dimension of the cross-section) of the frame elements and vectors of plasticity multipliers $\boldsymbol{\lambda}_j \geqslant 0$, $j \in J$ are the unknowns of the non-linear mathematical programming problem (33)–(37). Formulas (35) represent the complementary slackness conditions of mathematical programming [35]. The lower bound of the cross-sectional areas $A_{k,\min}$ is included in the construction constraints (36) $A_k \geqslant A_{k,\min}$. It is not difficult to introduce elastic displacements into the stiffness constraints (37) (see inequalities (21)). The limit moments $\mathbf{M}_0$ and influence matrixes $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\mathbf{G}$, $\mathbf{H}$ are related to the unknowns A_k, $k \in K$; the listed matrixes are recalculated during the solution of the problem (33)–(37). If the stiffness constraints are neglected, cyclic–plastic collapse of the frame occurs.

When only the bending moments M are taken into account in the frame calculation, the following mathematical model of frame volume minimization is obtained:

find

$$\min \quad \sum_k L_k A_k, \tag{38}$$

$$\text{subject to} \quad f_{\max} = \mathbf{M}_0 - \mathbf{G}\boldsymbol{\lambda} - \mathbf{M}_{e,\max} \geqslant 0,$$

$$f_{\min} = \mathbf{M}_0 + \mathbf{G}\boldsymbol{\lambda} + \mathbf{M}_{e,\min} \geqslant 0, \tag{39}$$

$$\boldsymbol{\lambda}_{\max}^{\mathrm{T}} f_{\max} = 0, \quad \boldsymbol{\lambda}_{\min}^{\mathrm{T}} f_{\min} = 0,$$

$$\boldsymbol{\lambda}_{\max} \geqslant 0, \quad \boldsymbol{\lambda}_{\min} \geqslant 0, \tag{40}$$

$$\boldsymbol{\lambda} = (\boldsymbol{\lambda}_{\max}, \boldsymbol{\lambda}_{\min})^{\mathrm{T}}, \tag{41}$$

$$A_k \geqslant A_{k,\min}, \quad k \in K, \tag{42}$$

$$\mathbf{u}_{r,\min} \leqslant \mathbf{u}_{r,\inf}, \mathbf{u}_{r,\sup} \leqslant \mathbf{u}_{r,\max}. \tag{43}$$

Extreme elastic bending moments $\mathbf{M}_{e,\max} = \boldsymbol{\alpha}_{\sup}\mathbf{F}_{\sup} + \boldsymbol{\alpha}_{\inf}\mathbf{F}_{\inf}$, $\mathbf{M}_{e,\min} = \boldsymbol{\alpha}_{\sup}\mathbf{F}_{\inf} + \boldsymbol{\alpha}_{\inf}\mathbf{F}_{\sup}$ are known in the problem (38)–(43). The unknowns are the cross-sectional areas A_k, $k \in K$ of the elements and the vectors of plasticity multipliers $\boldsymbol{\lambda}_{\max}$, $\boldsymbol{\lambda}_{\min}$.

In the case of a monotonically increasing load, $j = 1$ and conditions (34), (35) of all the discretized frame have the following form: $\mathbf{f} = \mathbf{M}_0 - \boldsymbol{\Phi}(\mathbf{G}\boldsymbol{\lambda} + \mathbf{S}_e) \geqslant \mathbf{0}$, $\boldsymbol{\lambda}^T[\mathbf{M}_0 - \boldsymbol{\Phi}(\mathbf{G}\boldsymbol{\lambda} + \mathbf{S}_e)] = 0$, $\boldsymbol{\lambda} \geqslant 0$. The stiffness constraints (37) of the frame are simplified: $\mathbf{u}_{r,\min} \leqslant \mathbf{H}\boldsymbol{\lambda} \leqslant \mathbf{u}_{r,\max}$. The scope of problem (33)–(37) is reduced and computer realization of the problem is simpler.

A brief description of the solution peculiarities of the volume minimization problem will follow. From the solution algorithm scheme (Fig. 4), it is possible to see that in the beginning both problems (33)–(43) are solved when the stiffness conditions (37) or (43) are changed into the constraints $\mathbf{u}_{r,\min} \leqslant \mathbf{H}\boldsymbol{\lambda} \leqslant \mathbf{u}_{r,\max}$ of the corresponding holonomic process. For instance, first the following simplified variant of the problem (38)–(43):

find

$$\min \quad \sum_k L_k A_k, \tag{44}$$

subject to (39)–(42) and

$$\mathbf{u}_{r,\min} \leqslant \mathbf{H}\boldsymbol{\lambda} \leqslant \mathbf{u}_{r,\max}. \tag{45}$$

is solved. After an optimal solution of the problem (44) and (45) is found, stricter stiffness constraints (43) are verified using displacement bounds $\tilde{\mathbf{u}}_{r,\inf}$, $\tilde{\mathbf{u}}_{r,\sup}$ or $\mathbf{u}_{r,\inf}$, $\mathbf{u}_{r,\sup}$. In the scheme of the solution algorithm of the volume minimization problem (Fig. 4), the stiffness conditions are related to the bounds $\mathbf{u}_{r,\inf}$, $\mathbf{u}_{r,\sup}$.

It should be noted that the numerical solution of the problems (33)–(43) is easier when the complementary slackness conditions are moved to the objective function. Then, for example, the objective function of the problem (38)–(43) has the following form:

$$\min\left(\sum_k L_k A_k + \boldsymbol{\lambda}_{\max}^T f_{\max} + \boldsymbol{\lambda}_{\min}^T f_{\min}\right).$$

6.2. Shakedown load optimization of frames

In the case of a variable repeated load, there is also the important problem of determining the limits of the load $\mathbf{F}_{\sup}$, $\mathbf{F}_{\inf}$, which is stated as follows: find the shakedown

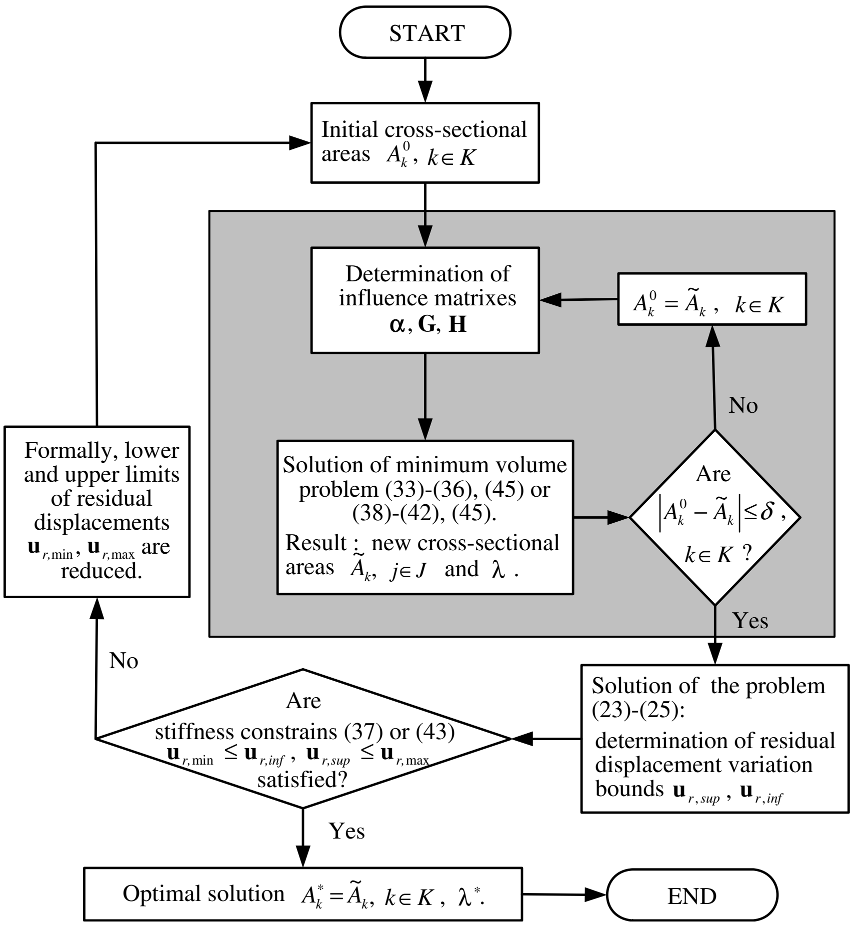


Fig. 4. Flowchart of the proposed solution algorithm.

load variational bounds $\mathbf{F}_{\sup}$, $\mathbf{F}_{\inf}$, satisfying the prescribed optimality criterion $\max\{\mathbf{T}_{\sup}^{\mathrm{T}}\mathbf{F}_{\sup} - \mathbf{T}_{\inf}^{\mathrm{T}}\mathbf{F}_{\inf}\}$, also the strength and stiffness requirements of the structure. Here $\mathbf{T}_{\sup}$, $\mathbf{T}_{\inf}$ are the optimality criterion weight coefficient vectors.

Then the mathematical model of the shakedown load optimization problem for frames reads:

find

$$\max \quad \left\{\mathbf{T}_{\sup}^{\mathrm{T}}\mathbf{F}_{\sup} - \mathbf{T}_{\inf}^{\mathrm{T}}\mathbf{F}_{\inf} - \sum_{j=1}^{p}\boldsymbol{\lambda}_j^{\mathrm{T}}[\mathbf{M}_0 - \boldsymbol{\Phi}(\mathbf{G}\boldsymbol{\lambda} + \mathbf{S}_{ej})]\right\}, \tag{46}$$

$$\text{subject to} \quad \mathbf{f}_j = \mathbf{M}_0 - \boldsymbol{\Phi}(\mathbf{G}\boldsymbol{\lambda} + \mathbf{S}_{ej}) \geqslant \mathbf{0}, \tag{47}$$

$$\boldsymbol{\lambda}_j \geqslant 0, \quad \boldsymbol{\lambda} = \sum_{j=1}^{p}\boldsymbol{\lambda}_j, \quad j \in J, \tag{48}$$

$$\mathbf{F}_{\sup} \geqslant 0, \quad -\mathbf{F}_{\inf} \geqslant 0; \tag{49}$$

$$\mathbf{u}_{r,\min} \leqslant \mathbf{u}_{r,\inf}, \quad \mathbf{u}_{r,\sup} \leqslant \mathbf{u}_{r,\max}. \tag{50}$$

The vector of limit bending moments $\mathbf{M}_0$ and the limits of the residual displacements $\mathbf{u}_{r,\min}$, $\mathbf{u}_{r,\max}$ are known in the problem (46)–(50). The optimal solution of the problem (46)–(50) is the vectors $\mathbf{F}_{\sup}^*$, $\mathbf{F}_{\inf}^*$ and $\boldsymbol{\lambda}_j^*$, $j \in J$.

7. Optimal shakedown design of trusses

7.1. Evaluation of bar stability

The yield conditions of a discretized truss read:

$$\mathbf{f}_{\max} = \mathbf{N}_0 - \mathbf{N}_r - \mathbf{N}_{e,\max} \geqslant 0, \tag{51}$$

$$\mathbf{f}_{\min} = \mathbf{N}_{0,cr} + \mathbf{N}_r + \mathbf{N}_{e,\min} \geqslant 0. \tag{52}$$

where $\mathbf{N}_{e,\max} = \boldsymbol{\alpha}_{\sup}\mathbf{F}_{\sup} + \boldsymbol{\alpha}_{\inf}\mathbf{F}_{\inf}$, $\mathbf{N}_{e,\min} = \boldsymbol{\alpha}_{\sup}\mathbf{F}_{\inf} + \boldsymbol{\alpha}_{\inf}\mathbf{F}_{\sup}$ are the vectors of the minimum and maximum values of the elastic axial forces. Here $\mathbf{N}_0 = (N_{0k})^{\mathrm{T}}$, $\mathbf{N}_{0,cr} = (N_{0k,cr})^{\mathrm{T}}$, $N_{0,k} = \sigma_{yk}A_k$, $N_{0,k,\ cr} = \varphi_k\sigma_{yk}A_k$, $k \in K$. The possible failure of bars under compression because of lost stability is evaluated by introducing the reduced limit axial force vector $\mathbf{N}_{0,cr}$ in the yield conditions (52). The components $N_{0,cr,k}$ of the vector $\mathbf{N}_{0,cr}$ are determined according to the recommendations of Eurocode 3:

$$N_{0,cr,k} = \varphi_k N_{0,k}, \quad k \in K, \tag{53}$$

$$\varphi_k = \frac{1}{\Phi_k + [\Phi_k^2 - \bar{\lambda}_k^2]^{0.5}}, \tag{54}$$

where $\Phi_k = 0.5(1 + a(\bar{\lambda}_k - 0.2) - \bar{\lambda}_k^2)$, $\bar{\lambda}_k = \frac{\lambda_k}{\lambda_{1k}}\sqrt{\beta_A} = \frac{\lambda_k}{\pi[E_k/\sigma_{y,k}]^{0.5}}\sqrt{\beta_A}$. Here $\sigma_{y,k}$ and E_k are the material yield limit and the modulus of elasticity of the kth bar; $\lambda_k = L_k/i_k$ is the bar slenderness, where i_k is the radius of gyration of the kth bar. In the case of a bar under pure compression $\beta_A = 1$, the value of the imperfection factor a depends on the shape of the cross-sections and the properties of the material used ($a = 0.21$ for hot rolled pipes). A possible failure because of loss of stability of the bar system is not evaluated when $\mathbf{N}_{0,cr} = \mathbf{N}_0$.

7.2. The problem of truss volume minimization

The minimum volume of a truss can be determined by solving the following problem:

find

$$\min \quad \sum_k L_kA_k + \boldsymbol{\lambda}_{\max}^{\mathrm{T}}[\mathbf{N}_0 - (\mathbf{G}\boldsymbol{\lambda} + \mathbf{N}_{e,\max})] + \boldsymbol{\lambda}_{cr}^{\mathrm{T}}[\mathbf{N}_{0,cr} + (\mathbf{G}\boldsymbol{\lambda} + \mathbf{N}_{e,\min})], \tag{55}$$

$$\text{subject to} \quad \mathbf{f}_{\max} = \mathbf{N}_0 - \mathbf{G}\boldsymbol{\lambda} - \mathbf{N}_{e,\max} \geqslant 0, \tag{56}$$

$$\mathbf{f}_{\min} = \mathbf{N}_{0,cr} + \mathbf{G}\boldsymbol{\lambda} + \mathbf{N}_{e,\min} \geqslant 0, \tag{57}$$

$$\boldsymbol{\lambda}_{\max} \geqslant 0, \quad \boldsymbol{\lambda}_{cr} \geqslant 0, \quad \boldsymbol{\lambda} = (\boldsymbol{\lambda}_{\max}, \boldsymbol{\lambda}_{cr})^{\mathrm{T}}, \tag{58}$$

$$A_k \geqslant A_{k,\min}, \quad k \in K, \tag{59}$$

$$\mathbf{u}_{r,\min} \leqslant \mathbf{u}_{r,\inf}, \quad \mathbf{u}_{r,\sup} \leqslant \mathbf{u}_{r,\max}. \tag{60}$$

Here the load variation bounds $\mathbf{F}_{\inf}$, $\mathbf{F}_{\sup}$ are prescribed, so in the mathematical model (55)–(60) the extreme forces $\mathbf{N}_{e,\max}$, $\mathbf{N}_{e,\min}$ are known. It is not difficult to introduce elastic displacements into the stiffness constraints (60) by applying formula (9). The unknowns of the problem (55)–(60) are the cross-sectional areas A_k, $k \in K$ of the truss elements and the vectors of plasticity multipliers $\boldsymbol{\lambda}_{\max}$, $\boldsymbol{\lambda}_{cr}$. The stiffness constraints (60), requiring the solution of problems (23)–(25), show that the main non-linear truss-optimization problem is not also a classical mathematical programming problem.

The minimum of the objective function (55) is obtained by neglecting the possible loss of bar stability if the factor of yield stress reduction is $\varphi_k = 1$ ($k \in K$) in the yield conditions (57) of the mathematical model (55)–(60). The minimum truss volume would be obtained according to the conditions of cyclic–plastic collapse if the stiffness constrains (60) were neglected.

7.3. Problem of load optimization

The mathematical model of the shakedown load optimization problem for trusses is based on the problem (46)–(50) and is stated as follows:

find

$$\max \quad \{\mathbf{T}_{\sup}^{\mathrm{T}}\mathbf{F}_{\sup} - \mathbf{T}_{\inf}^{\mathrm{T}}\mathbf{F}_{\inf} - \boldsymbol{\lambda}_{\max}^{\mathrm{T}}[\mathbf{N}_0 - (\mathbf{G}\boldsymbol{\lambda} + \mathbf{N}_{e,\max})] - \boldsymbol{\lambda}_{cr}^{\mathrm{T}}[\mathbf{N}_{0,cr} + (\mathbf{G}\boldsymbol{\lambda} + \mathbf{N}_{e,\min})]\}, \tag{61}$$

$$\text{subject to} \quad \mathbf{f}_{\max} = \mathbf{N}_0 - \mathbf{G}\boldsymbol{\lambda} - \mathbf{N}_{e,\max} \geqslant 0, \tag{62}$$

$$\mathbf{f}_{\min} = \mathbf{N}_{0,cr} + \mathbf{G}\boldsymbol{\lambda} + \mathbf{N}_{e,\min} \geqslant 0, \tag{63}$$

$$\boldsymbol{\lambda}_{\max} \geqslant 0, \quad \boldsymbol{\lambda}_{cr} \geqslant 0, \quad \boldsymbol{\lambda} = (\boldsymbol{\lambda}_{\max}, \boldsymbol{\lambda}_{cr})^{\mathrm{T}}, \tag{64}$$

$$\mathbf{F}_{\sup} \geqslant 0, \quad -\mathbf{F}_{\inf} \geqslant 0; \tag{65}$$

$$\mathbf{u}_{r,\min} \leqslant \mathbf{u}_{r,\inf}, \quad \mathbf{u}_{r,\sup} \leqslant \mathbf{u}_{r,\max}. \tag{66}$$

The limit axial force vectors $\mathbf{N}_0$, $\mathbf{N}_{0,cr}$ and the limits of residual displacements $\mathbf{u}_{r,\min}$, $\mathbf{u}_{r,\max}$ are known in the problem (61)–(66), the optimal solution of which is the vectors $\mathbf{F}^*_{\text{sup}}$, $\mathbf{F}^*_{\text{inf}}$ and $\boldsymbol{\lambda}^*_{\max}$, $\boldsymbol{\lambda}^*_{\text{cr}}$.

8. Numerical example

Proposed calculation technique is illustrated by example of minimization of three-storey frame (Fig. 5). The software M0opt1, which is created by authors, is based on Rosen project gradient method [24] and applied for solution of presented numerical example.

The three-storey frame shown in Fig. 5 is discretized by using equilibrium finite elements. Finite elements with six degrees of freedom are used for columns under bending and axial loading and finite elements with seven degrees of freedom are used for beam elements subjected to a distributed load with linear displacements of the central node (see Fig. 8). The later elements [36] exactly model the stress and strain field of the beams and allow the middle section displacements $u_{\text{r}10}$, $u_{\text{r}11}$,$u_{\text{r}12}$,$u_{\text{r}22}$,$u_{\text{r}23}$ of the beams to be computed directly. This creates the possibility of decreasing the number of unknowns in the optimization problem (33)–(37) and of obtaining information that is necessary for later analysis.

The frame is subjected to three independent load sets: horizontal concentrated forces $\mathbf{F}_1 = \{F_1^1, F_1^2, F_1^3, F_1^4, F_1^5, F_1^6, F_1^7\}$ acting on the nodes of the frame and vertical uniformly distributed forces $\mathbf{F}_2 = \{F_2^1, F_2^2\}$ acting on the roof beams and $\mathbf{F}_3 = \{F_3\}$ acting on the floor beams, respectively. Limits for the variations of the load are defined by the inequalities $\mathbf{F}_{1,\text{inf}} \leqslant \mathbf{F}_1 \leqslant \mathbf{F}_{1,\text{sup}}$, $\mathbf{F}_{2,\text{inf}} \leqslant \mathbf{F}_2 \leqslant \mathbf{F}_{2,\text{sup}}$ and $\mathbf{F}_{3,\text{inf}} \leqslant \mathbf{F}_3 \leqslant \mathbf{F}_{3,\text{sup}}$, where $\mathbf{F}_{1,\ \text{inf}} = \{-5.16, -6.06, -3.6, -7.8, -6.6, -6, -10.2\}$ kN, $\mathbf{F}_{1,\text{sup}} = \{10.2, 12.6, 7.8, 3.6, 3.36, 2.7, 5.16\}$ kN, $\mathbf{F}_{2,\text{inf}} = \{0, 0\}$, $\mathbf{F}_{2,\text{sup}} = \{2.52, 5.22\}$ kN/m, $\mathbf{F}_{3,\text{inf}} = \{0\}$ and $\mathbf{F}_{3,\text{sup}} = \{30\}$ kN/m.

The frame is made of steel with a modulus of elasticity $E = 21{,}000$ kN/cm^2 and a yield limit $\sigma_{\text{y}} = 23.5$ kN/cm^2. The cross-sections of the frame column, roof and floor

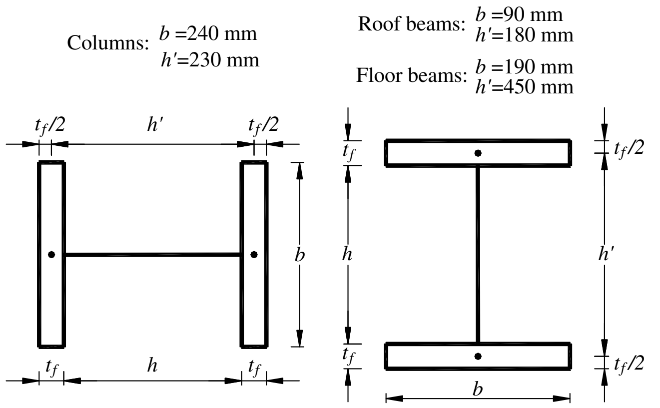


Fig. 6. Geometry of cross-sections.

beams are shown in Fig. 6. The parameters b and h' remain the same throughout the optimization process, only the thickness of the flanges varying. The initial flange thickness is taken as $t^0_{\text{f,col}} = 12$ mm for the frame columns, $t^0_{\text{f,roof beam}} = 8$ mm for the roof beams and $t^0_{\text{f,floor beam}} = 8$mm for the floor beams. Thus, the initial cross-sectional areas of the columns, roof and floor beams are $A^0_{\text{col}} = A^0_1 = A^0_2 = A^0_3 = A^0_4 = A^0_5 = A^0_6 = A^0_7 = A^0_8 = 57.6$ cm^2, $A^0_{\text{roof beam}} = A^0_9 = A^0_{10} = 14.4$ cm^2 and $A^0_{\text{floor beam}} = A^0_{11} = A^0_{12} = A^0_{13} = 57$ cm^2, respectively. The initial volume of structure is $V^0 = 279{,}540$ cm^3. The limit forces of the cross-sections are calculated according to

$$M_0 = \sigma_y \cdot b \cdot t \cdot h' = \sigma_y \cdot A \cdot \frac{h'}{2}, \quad N_0 = \sigma_y \cdot 2b \cdot t = \sigma_y \cdot A.$$

The initial limit forces of the columns are $M^0_{0,\text{col}} = 155.66$ kN m and $N^0_{0,\text{col}} = 1353.6$ kN, the limit forces of the roof and floor beams are $M^0_{0,\text{roof beam}} = 30.456$ kN m, $N^0_{0,\text{roof beam}} = 338.4$ kN and $M^0_{0,\text{floor beam}} = 301.388$ kN m, $N^0_{0,\text{floor beam}} = 1339.5$ kN; also $c_{\text{col}} = 0.115$, $c_{\text{floor beam}} = 0.09$, $c_{\text{roof beam}} = 0.225$.

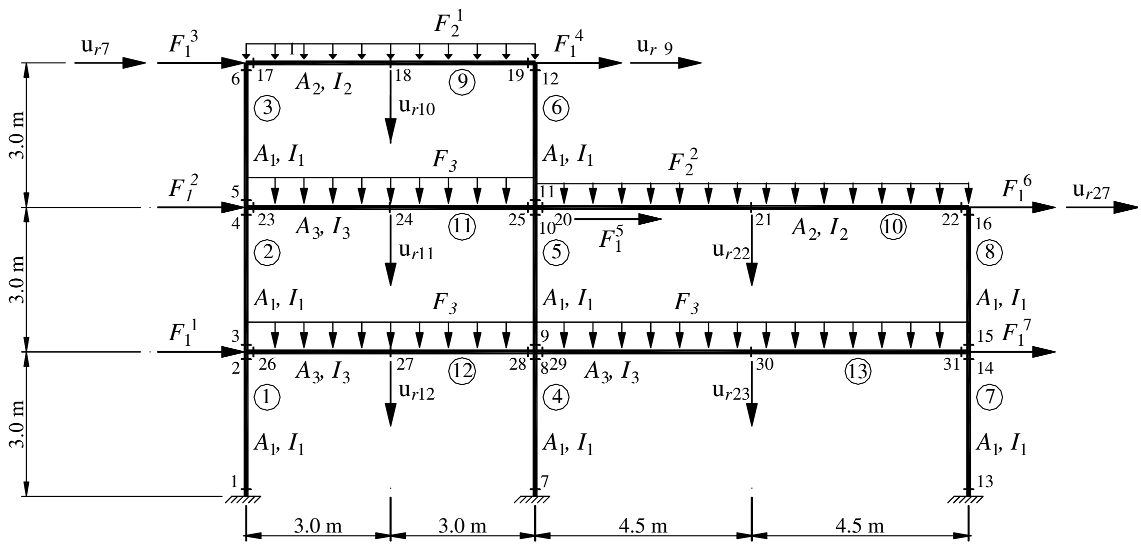


Fig. 5. Discretized frame.

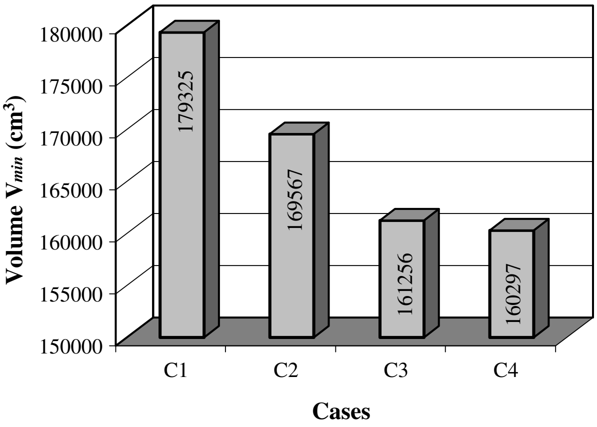


Fig. 7. Variation of frame minimal volume $V_{\min}$.

Table 1
Variation of the residual displacements u_{ri} of the beams

Cases	u_{r22} (mm)	u_{r23} (mm)	Location of the plastic strains
C1	**6.0**	**6.0**	7, 8, 14, 20, 22, 29
C2	**12.0**	**12.0**	8, 14, 15, 20, 22, 29
C3	**18.0**	**18.0**	7, 8, 14, 15, 20, 22, 29, 30
C4	**24.0**	14.48	7, 8, 14, 15, 20, 22, 29, 30

The main task is to determine the minimum volume of the adapted frame (Fig. 5) in the case when the vector of inner forces of the discretized frame is $\mathbf{S} = (\mathbf{M},\mathbf{N})^T = (M_1, M_2, M_3, \ldots, M_{31}, N_1, N_2, \ldots, N_{13})^T = (S_i)^T$, $i = 1,2, \ldots, n = 44$, i.e. both bending moments M and axial forces N are taken into account. In this case the frame volume minimization is performed according to the mathematical model (33)–(37). The unknowns are the cross-sectional areas of the frame columns and beams A_k, $k \in K$ and the vectors of plasticity multipliers $\boldsymbol{\lambda}_j$, $j = 1,2,\ldots,8$. Problem (33)–(37) was solved according to the sequence of operations shown in Fig. 4.

When the residual displacement constraints (37) are neglected, the following results were obtained for the frame: minimum volume $V_{\min} = 156{,}724\ \text{cm}^3$; residual displacements of beams $u_{r10} = 0.088$ mm, $u_{r11} = 0.36$ mm, $u_{r12} = 0.77$ mm, $u_{r22} = 51.46$ mm, $u_{r23} = 12.62$ mm; plastic strains appears in sections 7, 8, 14, 15, 20, 22, and 29 (Fig. 5).

The following residual displacement constraints were imposed for vertical displacements of beams u_{r22}, u_{r23} (Fig. 5), in four cases:

C1 $-6 \leqslant u_{r22} \leqslant 6,\ -6 \leqslant u_{r23} \leqslant 6;$
C2 $-12 \leqslant u_{r22} \leqslant 12,\ -12 \leqslant u_{r23} \leqslant 12;$
C3 $-18 \leqslant u_{r22} \leqslant 18,\ -18 \leqslant u_{r23} \leqslant 18;$
C4 $-24 \leqslant u_{r22} \leqslant 24,\ -24 \leqslant u_{r23} \leqslant 24.$

Units of displacement constraints are millimetres. The calculation results depending on prescribed limits is shown in Fig. 7 and Table 1.

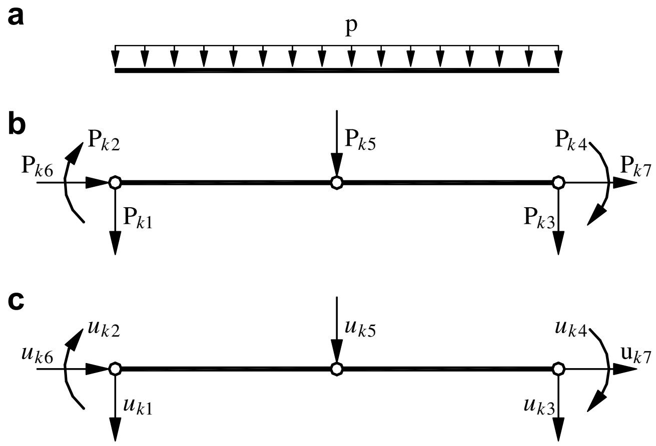


Fig. 8. Finite element subjected by distributed load with linear displacements of central node: (a) external load; (b) generalised forces; (c) nodal displacements.

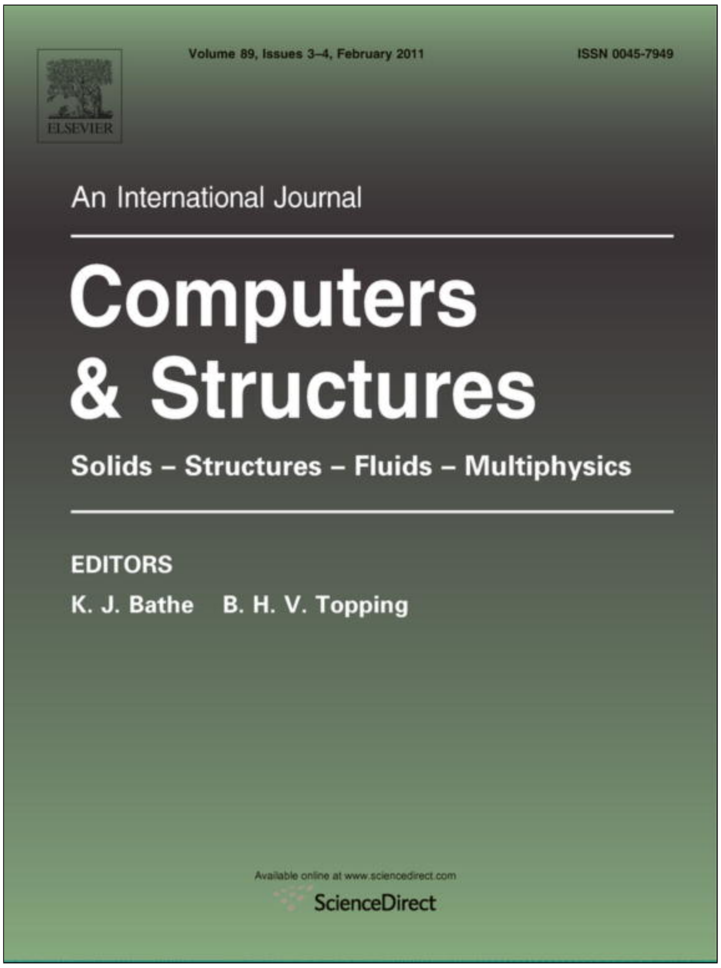
9. Conclusions

The formulation and solution of mathematical models for optimization problems in structural mechanics is just a first step in practical structural design, and also for structures at shakedown. An adapted structure is safe with respect to cyclic–plastic collapse but does not satisfy its serviceability requirements. Strength, stiffness and stability constraints should be included in the mathematical models of structure optimization. The determination of displacements is especially complicated if a variable repeated load is defined by the variational bounds. During the shakedown process the residual displacements vary non-monotonically as a result of the phenomenon of unloading cross-sections. The complementary slackness conditions of mathematical programming do not allow this physical phenomenon to be evaluated. Thus, the non-linear problems of volume minimization and shakedown load optimization are not traditional mathematical programming problems: while solving them, it is necessary to check the stiffness conditions, i.e. to determine the lower and upper bounds of the residual and elastic displacements.

References

[1] Atkočiūnas J, Merkevičiūtė D. Optimal shakedown design of bar systems: strength, stiffness and stability constraints. In: Proceedings of the seventh international conference on computational structures technology. Stirling: Civil-Comp Press; 2004. p. 19 [CD-ROM].
[2] Atkočiūnas J, Merkevičiūtė D. Optimal shakedown design of bar systems: strength, stiffness and stability constraints. In: Proceedings of the seventh international conference on computational structures technology. Stirling: Civil-Comp Press; 2004. p. 361–3.
[3] Dorosz S, König JA, Sawczuk A, Kowal Z, Seidel W. Deflections of elastic–plastic hyperstatic beams under cyclic loading. Arch Mech 1981;33(5):611–24.
[4] Rozvany GIN. Optimal design of flexural systems. Oxford: Pergamon Press; 1976.
[5] Kaneko L, Maier G. Optimum design of plastic structures under displacement's constraints. Comput Methods Appl Mech Eng 1981;27(3):369–92.

[6] Čyras A. Extremum principles and optimization problems for linearly strain hardening elastoplastic structures. Appl Mech 1986;22(4):89–96.
[7] Kaliszky S, Lógó J. Optimal plastic limit and shakedown design of bar structures with constraints on plastic deformation. Eng Struct 1997;19(1):19–27.
[8] Atkočiūnas J. Mathematical models of optimization problems at shakedown. Mech Res Commun 1999;26(3):319–26.
[9] Tin-Loi F. Optimum shakedown design under residual displacement constraints. Struct Multidisc Optim 2000;19(2):130–9.
[10] Kaliszky S, Lógó J. Plastic behaviour and stability constraints in the shakedown analysis and optimal design of trusses. Struct Multidisc Optim 2002;24(2):118–24.
[11] Ponter ARS. An upper bound to the small displacements of elastic perfectly plastic structures. J Appl Mech 1972;l39:959–63.
[12] Capurso M, Corradi L, Maier G. Bounds on deformations and displacements in shakedown theory. In: Materiaux et Structures sous Chargement Cyclique. Paris: Assoc. Amicale des Ingénieurs Anciens Elèves de l'E. N. P. C.; 1979.
[13] Polizzotto C. Upper bounds on plastic strains for elastic–perfectly plastic solids subjected to variable loads. Int J Mech Sci 1979;21(6): 317–27.
[14] Atkočiūnas J, Borkowski A, König JA. Improved bounds for displacements at shakedown. Comput Methods Appl Mech Eng 1981;28(3):365–76.
[15] König JA. Shakedown of elastic–plastic structures. Amsterdam: Elsevier; 1987.
[16] Zyckowski M. Combined loadings in the theory of plasticity. Warsaw: Polish Scientific Publishers (PWM); 1981.
[17] Stein E, Zhang G, Mahnken R. Shakedown analysis for perfectly plastic and kinematic hardening materials. In: Stein E, editor. CISM, progress in computational analysis of inelastic structures. New York: Springer; 1993.
[18] Lange-Hasen P. Comparative study of upper bound methods for the calculation of residual deformation after shakedown. Lyngby: Technical University of Denmark, Department of Structural Engineering and Materials; 1998.
[19] Mróz Z, Weichert D, Dorosz S, editors. Inelastic behavior of structures under variable loads. Dordrecht: Kluwer Academic Publishers; 1995.
[20] Weichert D, Maier G, editors. Inelastic behavior of structures under variable repeated loads. New York, Vienna: Springer; 2002.
[21] Casciaro R, Garcea G. An iterative method for shakedown analysis. Comput Methods Appl Mech Eng 2002;191(49–50):5761–92.
[22] Atkočiūnas J. Design of elastoplastic systems under repeated loading. Vilnius: Science and Encyclopaedia Publishers; 1994 [in Russian].
[23] Merkevičiūtė D, Atkočiūnas J. Incremental method for unloading phenomenon fixation at shakedown. Journal of Civil Engineering and Management 2003;IX(3): 178–91.
[24] Bazaraa MS, Shetty CM. Nonlinear programming theory and algorithms. New York, Chichester, Brisbane, Toronto: John Wiley; 1979.
[25] Borkowski A, Atkočiūnas J. Optimal design for cyclic loading. In: IUTAM, symposium on optimization in structural design. Berlin: Springer Verlag; 1975.
[26] Atkočiūnas J, Merkevičiūtė D. Kuhn–Tucker conditions and load optimization problem at shakedown. In: Burczynski T, Fedelinski P, Majchrzak E, editors. Proceedings of the 15th international conference on computer methods in mechanics. Gliwice: Silesian Technical University; 2003.
[27] Belytschko T. Plane stress shakedown analysis by finite elements. J Mech Sci 1972;14:619–25.
[28] Belytschko T, Liu WK, Moran B. Nonlinear finite elements for continua and structures. New York: John Wiley & Sons; 2000.
[29] Gallager RH. Finite element analysis: fundamentals. Englewood Cliffs: Prentice-Hall Inc.; 1975.
[30] König JA, Kleiber M. On a new method of shakedown analysis. Bull Acad Pol Sci, Ser Sci Tech 1978;XXVI(4):167–71.
[31] Cohn MZ, Maier G. Engineering plasticity by mathematical programming. New York: Pergamon Press; 1978.
[32] Atkočiūnas J. Compatibility equations of strains for degenerate shakedown problems. Comput Struct 1997;63(2):277–82.
[33] Koiter WT. General theorems for elastic–plastic solids. In: Sheddon IN, Hills R, editors. Progress in solid mechanics. Amsterdam: North Holland; 1960.
[34] Atkočiūnas J, Merkevičiūtė D, Venskus A, Nagevičius J. Mathematical models for optimal shakedown trusses design problems in case of moving load. Technol Econ Dev Econ 2007;XIII(2):93–9 [in Lithuanian].
[35] Ferris MC, Tin-Loi F. On the solution of a minimum weight elastoplastic problem involving displacement and complementarity constraints. Comput Methods Appl Mech Eng 1999;174:107–20.
[36] Kalanta S, Grigusevičius A. Formulation of framed structures equations by static and mixed methods. Journal of Civil Engineering and Management 2003; IX(Suppl. 2):100–12.


Volume 89, Issues 3–4, February 2011
ISSN 0045-7949
ELSEVIER
An International Journal
Computers & Structures
Solids – Structures – Fluids – Multiphysics
EDITORS
K. J. Bathe B. H. V. Topping
Available online at www.sciencedirect.com
ScienceDirect

Computers and Structures 89 (2011) 435–443

Contents lists available at ScienceDirect

Computers and Structures

journal homepage: www.elsevier.com/locate/compstruc

Optimal shakedown design of frames under stability conditions according to standards

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ARTICLE INFO

Article history:
Received 10 November 2009
Accepted 23 November 2010
Available online 21 December 2010

Keywords:
Optimal shakedown design
Frames
Stability
Energy principles
Mathematical programming

ABSTRACT

A shakedown-frame plastic moments minimization and load-optimization nonlinear mathematical model with strength, stiffness, and stability constraints is investigated. A methodology and algorithms for stability evaluation have been developed according to various standards (Eurocode 3 (EC3) and the Dutch NEN 6771) by integrating the MatrixFrame commercial software for the building industry and the nonlinear mathematical programming software developed by the authors. For other investigators, this work makes it possible to integrate the solutions of nonlinear programming problems (plastic state variables – residual forces and displacements) into their structural design software. Numerical examples of optimization of frame structures are presented.

1. Introduction

The paper considers elastic–plastic frames affected by a variable repeated load which is a system of forces that may vary independently within prescribed bounds. Usually variable repeated forces are not characterized by a loading history $\boldsymbol{F}(t)$, but only by time-independent lower and upper bounds on the forces $\boldsymbol{F}_{\sup}, \boldsymbol{F}_{\inf}$, ($\boldsymbol{F}_{\inf} \leqslant \boldsymbol{F}(t) \leqslant \boldsymbol{F}_{\sup}$).

Today the evaluation of stability conditions for optimization problems involving elastic–plastic frames remains a topical scientific problem [1]. For example, it is permitted to design elastic–plastic frames using the EC3 or NEN 6771 standards, but in these standards, the methodology and algorithms for stability evaluation of shakedown structures are not fully described. This situation influenced the choice of topic for this paper: the optimal shakedown design of frames subjected to variable repeated load under strength, stiffness, and stability constraints. The aspects of the optimal shakedown design of bar structures under strength and stiffness conditions have been investigated in detail in [2–12]. In this research, two types of problems are considered [13]. The first problem is the plastic moments minimization of the shakedown frame. The unknowns in this problem are the plastic moments $\boldsymbol{M}_0$. The plastic moment, $M_0 = \sigma_y W_{pl}$, is the principal characteristic of the bending element section (σ_y is the yield limit of the material and W_{pl} the plastic section modulus).

The second problem is the load-optimization problem for a frame subjected to variable repeated load. By solving the load-optimization problem, the maximal load-variation bounds $\boldsymbol{F}_{\inf}$ and $\boldsymbol{F}_{\sup}$ which ensure frame integrity and which satisfy the stiffness and stability requirements of the structure can be found.

The solution of frame-optimization problems at shakedown is complicated because the stress–strain state of dissipative systems depends on their loading history [14–18]. These difficult optimization problems can be solved by using extremum energy principles and the theory of mathematical programming [19]. This makes it possible to create a new iterative algorithm based on the Rosen project-gradient method [20,21]. Stability requirements for both optimization problems can be evaluated by integrating the MatrixFrame commercial software for the building industry with the nonlinear mathematical programming software developed by the authors. The part of the problem solution that is related to stability is transferred to the design software which implements the EC3 and NEN 6771 standards. The solution procedure is therefore iterative, in that the structural or load constraints of each ordinary iteration of the main optimization problem are calculated using the MatrixFrame design software. In the proposed methodology, the initial data for the MatrixFrame design software are replaced by the residual forces and residual displacements obtained from the solution of the optimization problem, i.e., the evaluation of the influence of plastic deformations. A criterion for an optimal solution is the convergence within the desired tolerance of the objective function of the main optimization problem. For other investigators, the methodology developed here makes it possible

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doi:10.1016/j.compstruc.2010.11.014

to integrate the solutions of nonlinear programming problems (plastic state variables: residual forces and displacements) into their structural design software.

This paper is an updated and revised version of the conference paper [1]. The paper was extended by detailed explanation of the proposed nonlinear optimization mathematical models and by the in-depth description of how the variable repeated load is expressed by the load combinations which occur in engineering practice.

Numerical examples for frames are presented. The results are valid if small displacements are assumed.

2. General mathematical models

The discrete model of the frame at shakedown consists of s equilibrium finite elements. The limit force S_{0k} ($k = 1, 2, \ldots, s$) is assumed constant in the whole finite element. The kth element has s_k nodal points. The approximated nodal forces of each element are the bending moments M and axial forces N. Generalised nodal force $\boldsymbol{S}_v = (M_l, N_l)^{\mathrm{T}}$, $l = 1, 2, \ldots, s_k$, $v = 1, 2, \ldots, \zeta$, where ζ is the total number of discrete model design sections. The nodal internal forces of each element are a combination of one vector of length n of discrete model forces, $\boldsymbol{S} = (\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_v, \ldots, \boldsymbol{S}_\zeta)^{\mathrm{T}} = (S_z)^{\mathrm{T}}$, and one vector of length n, $\boldsymbol{\Theta} = (\boldsymbol{\Theta}_1, \boldsymbol{\Theta}_2, \ldots, \boldsymbol{\Theta}_v, \ldots, \boldsymbol{\Theta}_\zeta)^{\mathrm{T}} = (\Theta_z)^{\mathrm{T}}$, $z = 1, 2, \ldots, n$. The degrees of freedom are m, corresponding to m displacement vectors $\boldsymbol{u}_e = (u_{e,1}, u_{e,2}, \ldots, u_{e,m})^{\mathrm{T}}$.

The load $\boldsymbol{F}(t)$ is characterized by time t and the independent variation bounds, $\boldsymbol{F}_{\sup} = (F_{1,\sup}, F_{2,\sup}, \ldots, F_{m,\sup})^{\mathrm{T}}$ and $\boldsymbol{F}_{\inf} = (F_{1,\inf}, F_{2,\inf}, \ldots, F_{m,\inf})^{\mathrm{T}}$, $(\boldsymbol{F}_{\inf} \leqslant \boldsymbol{F}(t) \leqslant \boldsymbol{F}_{\sup})$. The elastic displacements $\boldsymbol{u}_e(t)$ and the forces $\boldsymbol{S}_e(t)$ of the structure are determined using influence matrices of displacements and forces, $\boldsymbol{\beta} = (\boldsymbol{AKA}^{\mathrm{T}})^{-1}$, $\boldsymbol{\alpha} = \boldsymbol{KA}^{\mathrm{T}}\boldsymbol{\beta}$, respectively, where $\boldsymbol{u}_e(t) = \boldsymbol{\beta F}(t)$, $\boldsymbol{S}_e(t) = \boldsymbol{\alpha F}(t)$, $\boldsymbol{K} = \boldsymbol{D}^{-1}$. Here $\boldsymbol{A}$ is a coefficient matrix of equilibrium equations, $\boldsymbol{AS} = \boldsymbol{F}$, and $\boldsymbol{D}$ is a quasi-diagonal flexibility matrix. The residual displacements $\boldsymbol{u}_r$ and the forces $\boldsymbol{S}_r$ are related to the vector of plasticity multipliers $\boldsymbol{\lambda}$ by the influence matrices $\boldsymbol{H}$ and $\boldsymbol{G}$, where $\boldsymbol{u}_r = \overline{\boldsymbol{H}}\boldsymbol{\Theta}^{\mathrm{T}}\boldsymbol{\lambda} = \boldsymbol{H\lambda}, \boldsymbol{S}_r = \overline{\boldsymbol{G}}\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\lambda} = \boldsymbol{G\lambda}, \overline{\boldsymbol{H}} = \boldsymbol{\alpha}^T$, and $\overline{\boldsymbol{G}} = \alpha\boldsymbol{AK} - \boldsymbol{K}$. Here $\boldsymbol{\Phi}$ is the matrix of piecewise-linearized yield conditions, $\boldsymbol{\varphi}_j$. The number of all possible combinations $\boldsymbol{F}_j$ of load bounds $\boldsymbol{F}_{\sup}, \boldsymbol{F}_{\inf}$ is $p = 2^m$ ($\boldsymbol{F}_{\inf} \leqslant \boldsymbol{F}_j \leqslant \boldsymbol{F}_{\sup}$), where $\boldsymbol{S}_{ej} = \boldsymbol{\alpha F}_j$, $\boldsymbol{u}_{ej} = \boldsymbol{\beta F}_j$, $j = 1, 2, \ldots, p$. It is possible to evaluate directly, not only the variable repeated load $\boldsymbol{F}_j$, but also other loads $\boldsymbol{F}_c$ (for example a persistent load), additionally including them in combination j. The elastic forces $\boldsymbol{S}_{ec}$ and elastic displacements $\boldsymbol{u}_{ec}$ resulting from the loads $\boldsymbol{F}_c$ are calculated as $\boldsymbol{S}_{ec} = \boldsymbol{\alpha F}_c$, $\boldsymbol{u}_{ec} = \boldsymbol{\beta F}_c$.

The general mathematical models presented in Table 1 are the basis for the development of the mathematical optimization models of frames at shakedown which are considered in this paper.

In both plastic moments minimization and load optimization, the objective functions are described by Eqs. (1) and (6), respectively, where the vectors $\boldsymbol{L}$, $\boldsymbol{T}_{\sup}$, and $\boldsymbol{T}_{\inf}$ contain weighting coefficients. The yield conditions $\boldsymbol{\varphi}_j (j = 1, 2, \ldots, p)$ are given by Eqs. (2) and (7), respectively, where j is the number of all possible combinations $\boldsymbol{F}_j$ of load bounds $\boldsymbol{F}_{\sup}, \boldsymbol{F}_{\inf}$. The complementary slackness conditions of mathematical programming are given by Eqs. (3) and (8), respectively. Eqs. (4) and (9) are the respective constraints for the problem unknowns. The vectors $\boldsymbol{M}_{\max}$, $\boldsymbol{M}_{\min}$, $\boldsymbol{F}_{\max}$, and $\boldsymbol{F}_{\min}$ play a major role in stability evaluation. For further details on this topic, see Section 3. The stiffness constraints are given in Eqs. (5) and (10), respectively.

The optimal parameters for frame design using mathematical model (1)–(5) can be calculated when the yield limit σ_{yk} of the frame material, the lengths L_k of all elements k ($k = 1, 2, \ldots, s$), and the load-variation bounds $\boldsymbol{F}_{\sup}, \boldsymbol{F}_{\inf}$ are known. Depending on the cross-sectional shape, various yield conditions can be assumed.

Table 1
General mathematical models of optimization problems.

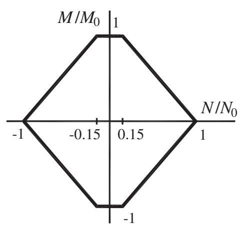
Plastic moments problem		Load-optimization problem	
Find		Find	
$\min \boldsymbol{L}^T\boldsymbol{M}_0$	(1)	$\max(\boldsymbol{T}_{\sup}^T\boldsymbol{F}_{\sup} - \boldsymbol{T}_{\inf}^T\boldsymbol{F}_{\inf})$	(6)
Subject to		Subject to	
$\boldsymbol{\varphi}_j = \boldsymbol{M}_0 - \boldsymbol{\Phi}(\boldsymbol{G\lambda} + \boldsymbol{S}_{ej} + \boldsymbol{S}_{ec}) \geq \boldsymbol{0}$	(2)	$\boldsymbol{\varphi}_j = \boldsymbol{M}_0 - \boldsymbol{\Phi}(\boldsymbol{G\lambda} + \boldsymbol{S}_{ej} + \boldsymbol{S}_{ec}) \geq 0$	(7)
$\boldsymbol{\lambda}_j^T\boldsymbol{\varphi}_j = 0, \boldsymbol{\lambda}_j \geq \boldsymbol{0}, \boldsymbol{\lambda} = \sum_j \boldsymbol{\lambda}_j, \quad j = 1, 2, \ldots, p$	(3)	$\boldsymbol{\lambda}_j^T\boldsymbol{\varphi}_j = 0, \boldsymbol{\lambda}_j \geq \boldsymbol{0}, \boldsymbol{\lambda} = \sum_j \boldsymbol{\lambda}_j, \quad j = 1, 2, \ldots, p$	(8)
$\boldsymbol{M}_{min} \leq \boldsymbol{M}_0 \leq \boldsymbol{M}_{max}$	(4)	$\boldsymbol{0} \leq \boldsymbol{F}_{sup} \leq \boldsymbol{F}_{max}, \boldsymbol{F}_{min} \leq \boldsymbol{F}_{inf} \geq 0$	(9)
$\boldsymbol{u}_{min} \leq (\boldsymbol{H\lambda} + \boldsymbol{u}_{ej} + \boldsymbol{u}_{ec}) \leq \boldsymbol{u}_{max}$	(5)	$\boldsymbol{u}_{min} \leq (\boldsymbol{H\lambda} + \boldsymbol{u}_{ej} + \boldsymbol{u}_{ec}) \leq \boldsymbol{u}_{max}$	(10)

This paper focuses on yield conditions for rolled I-beam steel sections (Fig. 1).

The relation $c_k = \frac{M_{0k}}{N_{0k}}, k \in K$ should be determined in advance. The limit moment, $M_{0k} = \sigma_{yk}W_{pl,k} = \xi(\sigma_{yk}, A_k)$, and the limit axial force, $N_{0k} = \sigma_{yk}A_k$, of the element are functions of the cross-sectional area, A_k, and the yield limit of the material, σ_{yk}. It is usually true that one or two specific dimensions of the cross-section (for instance, the flange thickness t_f and the web thickness t_w of the I-beam cross-section, while the width of the flange b and the height h are fixed); see Examples 1 and 2 can participate in the functional relation $M_{0k} = \xi(\sigma_{yk}, A_k)$. The limit moments M_{0k} of the frame elements and the vectors of plasticity multipliers $\boldsymbol{\lambda}_j \geqslant 0$, $j = 1, 2, \ldots, p$ are the unknowns of the nonlinear mathematical programming problem (1)–(5). The structural requirements for the frames, $\boldsymbol{M}_{\min}$ and $\boldsymbol{M}_{\max}$, are given by conditions (4). The limit moments $\boldsymbol{M}_0$ and the influence matrices $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{G}$, $\boldsymbol{H}$ are related to the A_k, $k = 1, 2, \ldots, s$; these matrices are recalculated during the solution of problem (1)–(5). If the stiffness and stability constraints are neglected, the frame will approach, but not reach, the point of cyclic-plastic collapse. Mathematical models of shakedown structures where cyclic-plastic collapse (incremental or alternating plasticity) occurs are described in [11]. The optimal solution of problem (1)–(5) consists of the vectors $\boldsymbol{M}_0^*$ and $\lambda_j^*, j = 1, 2, \ldots, p$.

In the case of variable repeated load, the problem of determining the load-variation bounds $\boldsymbol{F}_{\sup}, \boldsymbol{F}_{\inf}$ for problem (6)–(10) is also important. This problem can be stated as follows: find the shakedown load-variation bounds $\boldsymbol{F}_{\sup}, \boldsymbol{F}_{\inf}$, which satisfy the prescribed optimality criterion, $\max(\boldsymbol{T}_{\sup}^T\boldsymbol{F}_{\sup} - \boldsymbol{T}_{\inf}^T\boldsymbol{F}_{\inf})$, and also the strength, stiffness, and stability requirements of the structure. The vector of limit bending moments $\boldsymbol{M}_0$ and the limits $\boldsymbol{u}_{\min}$, $\boldsymbol{u}_{\max}$ of the total displacements $\boldsymbol{u} = \boldsymbol{u}_r + \boldsymbol{u}_{ej} + \boldsymbol{u}_{ec}$ are known from problem (6)–(10). The optimal solution of this problem consists of the vectors $\boldsymbol{F}_{\sup}^*, \boldsymbol{F}_{\inf}^*$, and $\lambda_j^*, j = 1, 2, \ldots, p$.

A rearrangement of mathematical models (1)–(5) and (6)–(10) for purposes of computer implementation is presented in Table 2.



$$\Phi_v = \begin{bmatrix} 0.85 & c_k \\ -0.85 & c_k \\ 0.85 & -c_k \\ -0.85 & -c_k \\ 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad v = 1, 2, \ldots, \zeta$$

Fig. 1. Linear yield conditions.

In mathematical models (11)–(15) and (16)–(20), the complementary slackness conditions given in Eqs. (3) and (8), $\lambda_j^T \varphi_j = 0$, are moved to the objective functions given in Eqs. (11) and (16). This rearrangement is made because the optimal solution gives $\lambda_j^T \varphi_j = 0$ and because of the Lagrangian relaxation principle, which allows placing constraints into an objective function. If the complementary slackness condition is part of the objective function, the optimization process is considerably faster, because the condition $\lambda_j^T \varphi_j = 0$ is satisfied, not during ordinary iteration, but only when the optimal solution is reached.

3. Stability evaluation

The stability of mathematical models (11)–(15) and (16)–(20) is evaluated using the structural restrictions given by Eqs. (14) and (19), respectively, which are calculated according to the stability requirements of the EC3 or NEN 6771 standards (or even another standard). Various standards have been implemented in commercial software that is available to meet the needs of designers. The authors of this paper have used the MatrixFrame building-industry software, version 4.1, for stability evaluation. Stability checks can be performed in MatrixFrame for both standards mentioned. In the case of EC3, the buckling resistance of members is calculated using equations given in Table 3. In the case of NEN 6771, the stability check is performed using equations given in Table 4. An element k meets the stability requirements when the maximum stability unity check (UC_k) calculated using the equations in the standard is less than or equal to unity. UC is the ratio of the design value to the design resistance.

Frame plastic moments minimization is performed using mathematical model (11)–(15) in an iterative manner (Fig. 2).

Step 1. The influence matrices $\boldsymbol{\alpha}^0$, $\boldsymbol{\beta}^0$, $\boldsymbol{G}^0$, $\boldsymbol{H}^0$, and the coefficients $c_k^0, k = 1, 2, \ldots, s$ of the yield conditions are determined for the assumed initial cross-sectional areas $A_k^0, k = 1, 2, \ldots, s$. Constraints (14) for the problem variables M_{0k} are $M_{0,k,\min} = 0 \leqslant M_{0,k} \leqslant M_{0,k,\max} = \infty$ (the only constraint on variable $M_{0,k}$ sign is applied).

Step 2. The problem described in Eqs. (11)–(15) is solved, and the new distribution of limit moments $M_{0k}^*, k = 1, 2, \ldots, s$, is determined. The selection of new sections can be performed in two ways: by changing the cross-sectional dimensions (continuous optimization) or by selecting a set of new sections from an available assortment of manufactured cross-sections using the criterion $W_{pl}^* \geqslant M_{0k}^* / \sigma_{yk}$ (discrete optimization).

Step 3. Plastic state variables – residual forces $\mathbf{S}_r$, and displacements $\mathbf{u}_r$ are introduced into the MatrixFrame stability calculation. If the maximal stability $UC_k > 1$, $k = 1, 2, \ldots, s$, then by changing the cross-sectional dimensions or selecting a new section from an available set, a new cross-section is found which has the property $UC_k \leqslant 1$. In this case, $M_{0k,min}$ has been determined. This means that, in the next iteration, the limit moment M_{0k} should be greater or equal to $M_{0k,min}$.

Step 4. New influence matrices $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{G}$, $\boldsymbol{H}$, and new coefficients c_k, $k = 1, 2, \ldots, s$, are determined for the cross-sections with areas A_k obtained in Step 2.

Step 5. Problem (11)–(15) is solved again using the recalculated matrices $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{G}$, $\boldsymbol{H}$, the recalculated coefficients c_k, and the new $M_{0k,min}$ obtained in Step 3.

Step 6. Steps 3–5 are repeated until the cross-sectional areas A_k obtained in two consecutive steps do not differ by more than a specified tolerance and the stability requirements are satisfied.

The stability requirements for all elements $k = 1, 2, \ldots, s$, are evaluated in Step 3 by finding cross-sections $A_k(M_{0k,\min})$ which satisfy the requirement that $UC_k \leqslant 1$.

The frame load optimization is performed using mathematical model (16)–(20), also in an iterative manner (Fig. 3).

Step 1. Problem (16)–(20) is solved, and the new vectors of load-variation bounds $\boldsymbol{F}_{\sup}, \boldsymbol{F}_{\inf}$ are determined. Constraints (19) on the problem variables $\boldsymbol{F}_{\sup}, \boldsymbol{F}_{\inf}$ are $0 \leqslant \boldsymbol{F}_{\sup} \leqslant \boldsymbol{F}_{\max} = \infty$, $\boldsymbol{F}_{\min} = -\infty \leqslant \boldsymbol{F}_{\inf} \leqslant 0$ (the only constraints on variables $\boldsymbol{F}_{\sup}$ and $\boldsymbol{F}_{\inf}$ sign are applied).

Step 2. Plastic state variables – residual forces $\mathbf{S}_r$ and displacements $\mathbf{u}_r$ are introduced into the MatrixFrame stability calculation. If the maximal stability $UC_k > 1$, $k = 1, 2, \ldots, s$, then by changing the load domain $\boldsymbol{F}_j$, a load domain is found that ensures that $UC_k \leqslant 1$. In this case, $\boldsymbol{F}_{\max}$ and $\boldsymbol{F}_{\min}$ have been found. This means that in the next iteration, the load-variation bounds $\boldsymbol{F}_{\sup}$ and $\boldsymbol{F}_{\inf}$ cannot exceed the load-variation bounds $\boldsymbol{F}_{\max}$ and $\boldsymbol{F}_{\min}$ which satisfy the stability requirements.

Step 3. Problem (16)–(20) is solved again using the load-variation bounds $\boldsymbol{F}_{\max}$ and $\boldsymbol{F}_{\min}$ obtained in Step 2.

Step 4. Steps 2 and 3 are repeated until the load-variation bounds $\boldsymbol{F}_{\sup}$ and $\boldsymbol{F}_{\inf}$ obtained in two consecutive steps do not differ by more than a specified tolerance and the stability requirements are satisfied.

The stability requirements for all elements $k = 1, 2, \ldots, s$, are evaluated in Step 2 by finding load-variation bounds $\boldsymbol{F}_{\max}$ and $\boldsymbol{F}_{\min}$ that satisfy the requirement that $UC_k \leqslant 1$.

Table 2
Mathematical models used in the computer implementation.

Plastic moments problem		Load-optimization problem	
Find		Find	
$min(\boldsymbol{L}^T\boldsymbol{M}_0 + \lambda_j^T\boldsymbol{\varphi}_j)$	(11)	$max(\boldsymbol{T}_{sup}^T\boldsymbol{F}_{sup} - \boldsymbol{T}_{inf}^T\boldsymbol{F}_{inf} - \lambda_j^T\boldsymbol{\varphi}_j)$	(16)
Subject to		Subject to	
$\boldsymbol{\varphi}_j = \boldsymbol{M}_0 - \boldsymbol{\Phi}(\boldsymbol{G}\boldsymbol{\lambda} + \boldsymbol{S}_{ej} + \boldsymbol{S}_{ec}) \geq \boldsymbol{0}$	(12)	$\boldsymbol{\varphi}_j = \boldsymbol{M}_0 - \boldsymbol{\Phi}(\boldsymbol{G}\boldsymbol{\lambda} + \boldsymbol{S}_{ej} + \boldsymbol{S}_{ec}) \geq 0$	(17)
$\lambda_j \geq 0, \boldsymbol{\lambda} = \sum_j \lambda_j, \quad j = 1, 2, \ldots, p$	(13)	$\lambda_j \geq 0, \boldsymbol{\lambda} = \sum_j \lambda_j, \quad j = 1, 2, \ldots, p$	(18)
$\boldsymbol{M}_{min} \leq \boldsymbol{M}_0 \leq \boldsymbol{M}_{max}$	(14)	$\boldsymbol{0} \leq \boldsymbol{F}_{sup} \leq \boldsymbol{F}_{max}, \boldsymbol{F}_{min} \leq \boldsymbol{F}_{inf} \leq 0$	(19)
$\boldsymbol{u}_{min} \leq (\boldsymbol{H}\lambda + \boldsymbol{u}_{ej} + \boldsymbol{u}_{ec}) \leq \boldsymbol{u}_{max}$	(15)	$\boldsymbol{u}_{min} \leq (\boldsymbol{H}\lambda + \boldsymbol{u}_{ej} + \boldsymbol{u}_{ec}) \leq \boldsymbol{u}_{max}$	(20)

4. Numerical examples

4.1. Introduction to examples

An Example 1 of the plastic moments minimization problem (11)–(15) and Example 2 of the load-optimization problem (16)–(20) illustrate the proposed calculation technique. The convex nonlinear optimization software modules M0opt1 and MaxFopt1 were used for the first and second problems, respectively. They are developed by the authors and are based on the Rosen project-gradient method [21] and are used here to obtain a solution of the numerical example under study. For stability evaluation, the MatrixFrame software for the building industry is used. Both examples are applied to a two-story frame (Fig. 4). The frame is subjected to two sets of independent loads: the horizontal, concentrated forces $\boldsymbol{F}_1 = \{F_1^1, F_1^2, F_1^3, F_1^4, F_1^5\}$ acting on the nodes of the frame, and the vertical, uniformly distributed forces $\boldsymbol{F}_2 = \{F_2^1, F_2^2\}$ acting on the roof beams (6, 7, 8, 9). A permanent load $\boldsymbol{F}_c = 117$ kN/m acts on the floor beams (10, 11) The limits of load variations are defined by the inequalities $\boldsymbol{F}_{1,\inf} \leqslant \boldsymbol{F}_1 \leqslant \boldsymbol{F}_{1,\sup}$,

Table 3
Stability evaluation formulas according to EC3 standard.

$\frac{N_{Ed}}{N_{b,Rd}} \leqslant 1.0,$	(EC3#6.46)
$\frac{M_{Ed}}{M_{b,Rd}} \leqslant 1.0,$	(EC3#6.54)
$\frac{N_{Ed}}{\frac{\chi_y N_{Rk}}{\gamma_{M1}}} + k_{yy}\frac{M_{y,Ed}+\Delta M_{y,Ed}}{\chi_{LT}\frac{M_{y,Rk}}{\gamma_{M1}}} + k_{yz}\frac{M_{z,Ed}+\Delta M_{z,Ed}}{\frac{M_{z,Rk}}{\gamma_{M1}}} \leqslant 1,$	(EC3#6.61)
$\frac{N_{Ed}}{\frac{\chi_z N_{Rk}}{\gamma_{M1}}} + k_{zy}\frac{M_{y,Ed}+\Delta M_{y,Ed}}{\chi_{LT}\frac{M_{y,Rk}}{\gamma_{M1}}} + k_{zz}\frac{M_{z,Ed}+\Delta M_{z,Ed}}{\frac{M_{z,Rk}}{\gamma_{M1}}} \leqslant 1,$	(EC3#6.62)
N_{Ed}	the design values of the compression force
$N_{b,Rd}$	the design buckling resistance of the compression member
M_{Ed}	the design value of the moment
$M_{b,Rd}$	the design buckling resistance moment
$M_{y,Ed}, M_{z,Ed}$	the maximum moments about the $y-y$ and $z-z$ axis along the member, respectively
$\Delta M_{y,Ed}, \Delta M_{z,Ed}$	the moments due to the shift of the centroidal axis, according to EC3#6.2.9.3 for class 4 sections, see EC3# Tables 6 and 7
χ_y, χ_z	the reduction factors due to flexural buckling from EC3#6.3.1
N_{Rk}	the characteristic resistance to normal force of the critical cross section
$M_{y,Rk}, M_{z,Rk}$	the characteristic moments resistance of the critical cross section about the $y-y$ and $z-z$ axis, respectively
χ_{LT}	the reduction factor due to lateral torsional buckling from EC3#6.3.2
$k_{yy}, k_{yz}, k_{zy}, k_{zz}$	the interaction factors
γ_{M1}	partial factor for resistance of members to instability assessed by member checks

$\boldsymbol{F}_{2,\inf} \leqslant \boldsymbol{F}_2 \leqslant \boldsymbol{F}_{2,\sup}$. It is noteworthy that the load combinations which occur in engineering practice can be modeled as separate cases of variable repeated load. The number of all possible combinations $\boldsymbol{F}_j$ of load bounds $\boldsymbol{F}_{\sup}, \boldsymbol{F}_{\inf}$ in the current example is $p = 2^2 = 4$ [20]. The load domain can be described using four load combinations:

(1) $\boldsymbol{F}_{1,\sup} + \boldsymbol{F}_{2,\sup} + \boldsymbol{F}_c$;
(2) $\boldsymbol{F}_{1,\sup} + \boldsymbol{F}_{2,\inf} + \boldsymbol{F}_c$;
(3) $\boldsymbol{F}_{1,\inf} + \boldsymbol{F}_{2,\sup} + \boldsymbol{F}_c$;
(4) $\boldsymbol{F}_{1,\inf} + \boldsymbol{F}_{2,\inf} + \boldsymbol{F}_c$.

The load combinations which occur in engineering practice can be described by introducing additional multipliers:

(1) $k_{11}\boldsymbol{F}_{1,\sup} + k_{12}\boldsymbol{F}_{2,\sup} + k_{13}\boldsymbol{F}c$;
(2) $k_{21}\boldsymbol{F}_{1,\sup} + k_{22}\boldsymbol{F}_{2,\inf} + k_{23}\boldsymbol{F}_c$;
(3) $k_{31}\boldsymbol{F}_{1,\inf} + k_{32}\boldsymbol{F}_{2,\sup} + k_{33}\boldsymbol{F}_c$;
(4) $k_{41}\boldsymbol{F}_{1,\inf} + k_{42}\boldsymbol{F}_{2,\inf} + k_{43}\boldsymbol{F}_c$,

where the values of the multipliers (the coefficients of each load combination) $k_{11}, k_{12}, \ldots, k_{43}$ and the load-variation bounds can be determined by the requirements of the various standards. For example, if $\boldsymbol{F}_1$ represents wind load, $\boldsymbol{F}_2$ snow load, and $\boldsymbol{F}_c$ permanent load, then the load bounds are: $\boldsymbol{F}_{1,\inf}$ = wind from right (WFR), $\boldsymbol{F}_{1,\sup}$ = wind from left (WFL), $\boldsymbol{F}_{2,\inf}$ = snow from bottom (SFB, included to complete the formal description, but cannot occur in reality), and $\boldsymbol{F}_{2,\sup}$ = snow from top (SFT). In this paper, the

Table 4
Stability evaluation formulas according to NEN6771 standard.

$\frac{N_{c;s;d}}{\omega_{z;buc}N_{c;u;d}} \leqslant 1$	(NEN6771#12.1-1a)
$\frac{N_{c;s;d}}{\omega_{y;buc}N_{c;u;d}} \leqslant 1,$	(NEN6771#12.1-1b)
$\frac{M_{y;max;s;d}}{\omega_{kip}M_{y;u;d}} \leqslant 1,$	(NEN6771#12.2-3)
$\frac{N_{c;s;d}}{N_{c;u;d}} + \frac{n_y}{n_y - 1}\frac{M_{y;\mathrm{equ};s;d} + F_{y;\mathrm{tot};s;d}e_y^*}{\omega_{kip}M_{y;u;d}} + \frac{n_z}{n_z - 1}\frac{\chi_y M_{z;\mathrm{equ};s;d}}{M_{z;u;d}} \leqslant 1$,	(NEN6771#12.3-1)
$\frac{N_{c;s;d}}{N_{c;u;d}} + \frac{n_y}{n_y - 1}\frac{\chi_z M_{y;\mathrm{equ};s;d}}{\omega_{kip}M_{y;u;d}} + \frac{n_z}{n_z - 1}\frac{M_{z;\mathrm{equ};s;d} + F_{z;\mathrm{tot};s;d}e_z^*}{M_{z;u;d}} \leqslant 1,$	(NEN6771#12.3-2)
$N_{c;s;d}$	the design values of the compression force
$N_{c;u;d}$	the reduction factors due to flexural buckling from
$\omega_{z;buc}, \omega_{y;buc}$	NEN6771# 12.1.1.4
$M_{y;\mathrm{max};s;d}$	the design value of the moment
$M_{y;u;d}$	the design buckling resistance moment
ω_{kip}	the reduction factor due to lateral torsional buckling
n_y, n_z	the proportionality coefficients
$M_{y;\mathrm{equ};s;d}, M_{y;\mathrm{equ};s;d}$	the equivalent moments about the $y - y$ and $z - z$ axis along the member, respectively
$F_{y;\mathrm{tot};s;d}, F_{z;\mathrm{tot};s;d}$	the values of the compression load
e_y^*, e_z^*	excentricities about the $y - y$ and $z - z$, respectively
χ_y, χ_z	the coefficients depending on the classification of the structure

distributed wind-load action is replaced by a set of concentrated equivalent loads, $\boldsymbol{F}_1 = \{F_1^1, F_1^2, F_1^3, F_1^4, F_1^5\}$. The numerical values of the load bounds are determined according to the Eurocode 1 standard. According to this standard, the load domain can be expressed as follows:

(1) $k_{11}WFL + k_{12}SFT + k_{13}\boldsymbol{F}_c$;
(2) $k_{21}WFL + k_{22}SFB + k_{23}\boldsymbol{F}_c$;
(3) $k_{31}WFR + k_{32}SFT + k_{33}\boldsymbol{F}_c$;
(4) $k_{41}WFR + k_{42}SFB + k_{43}\boldsymbol{F}_c$.

If external influences are incompatible (for example, snow and wind), then they can be easily excluded from the load combination by setting the corresponding multipliers to zero. In the current example, all multipliers $k_{11}, k_{12}, \ldots, k_{43}$ are equal to unity.

The vector of inner forces of the discretized frame is $\boldsymbol{S} = (\boldsymbol{M}, \boldsymbol{N})^T = (M_1, M_2, M_3, \ldots, M_{28}, N_1, N_2, \ldots, N_{11})^{\mathrm{T}} = (S_z)^T$, $z = 1, 2, \ldots, n = 39$, i.e., when both bending moments M and axial forces N are taken into account. The frame is made of steel, with a modulus of elasticity $E = 210$ GPa and a yield limit $\sigma_y = 235$ MPa. The material is elastic–perfectly plastic. The cross-sections of the frame columns, roof, and floor beams are shown in Fig. 5. The upper bound of total displacements constraints $\boldsymbol{u}_{\max}$ are chosen according to ratio $L_k/\delta_{\max}$ where L_k is the length of the kth element (beam), $\delta_{\max}$ is the value related to building type and is specified in national standards; in the paper $\delta_{\max} = 200$ is assumed. The

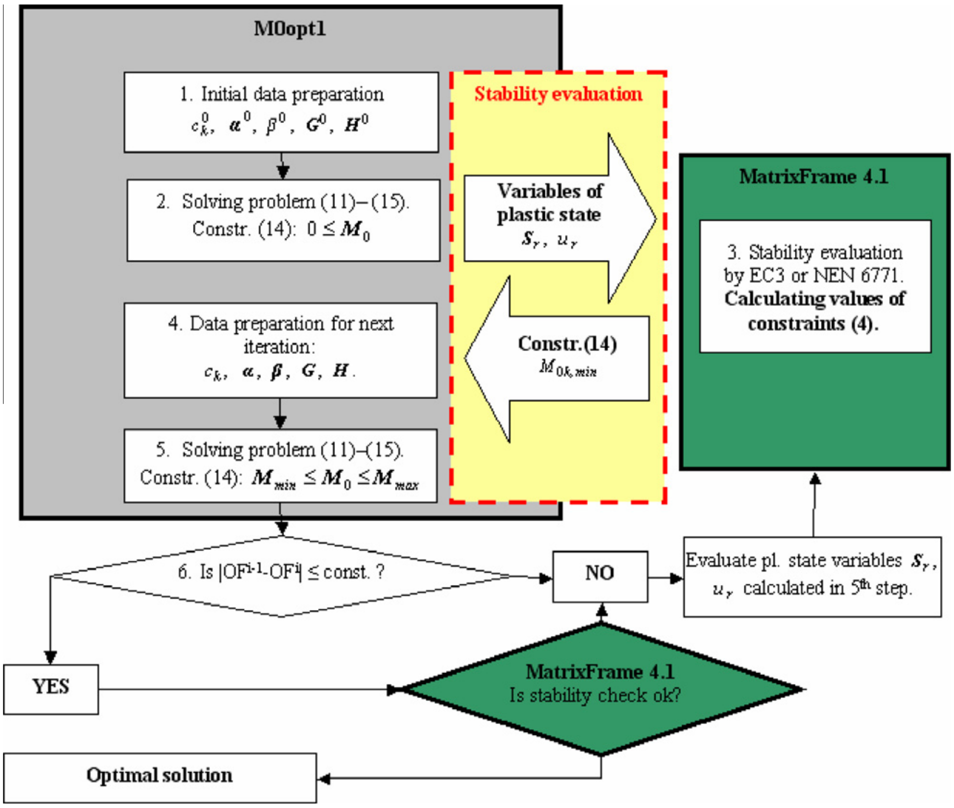


Fig. 2. Flowchart of the proposed solution algorithm for the volume-minimization problem.

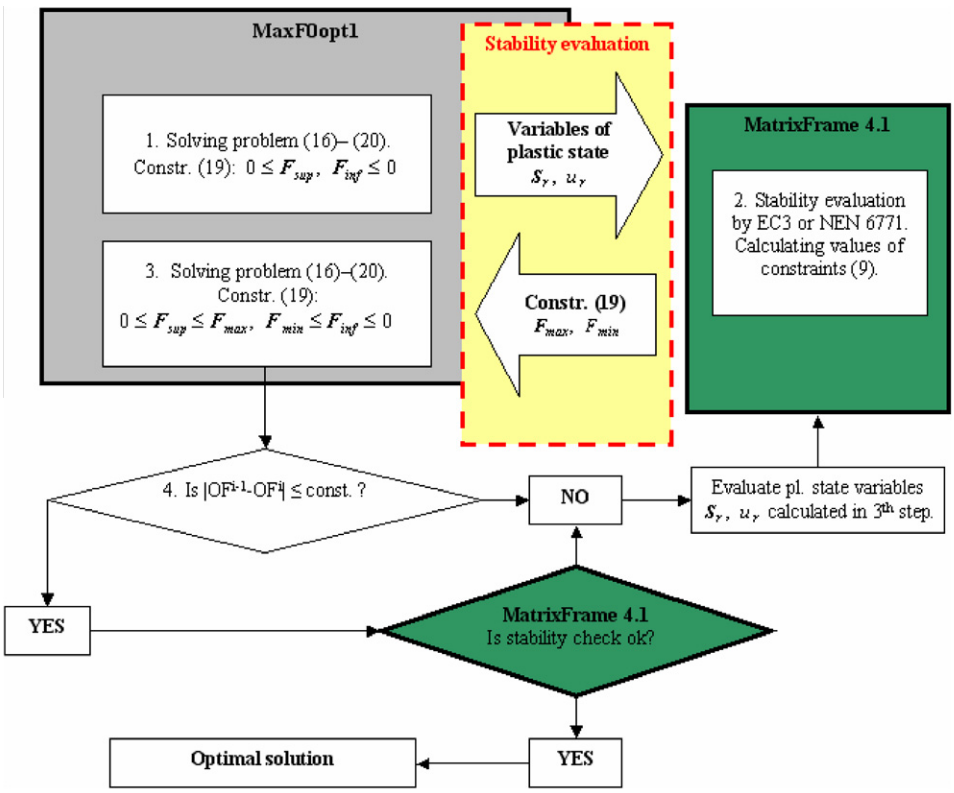


Fig. 3. Flowchart of the proposed solution algorithm for the load-optimization problem.

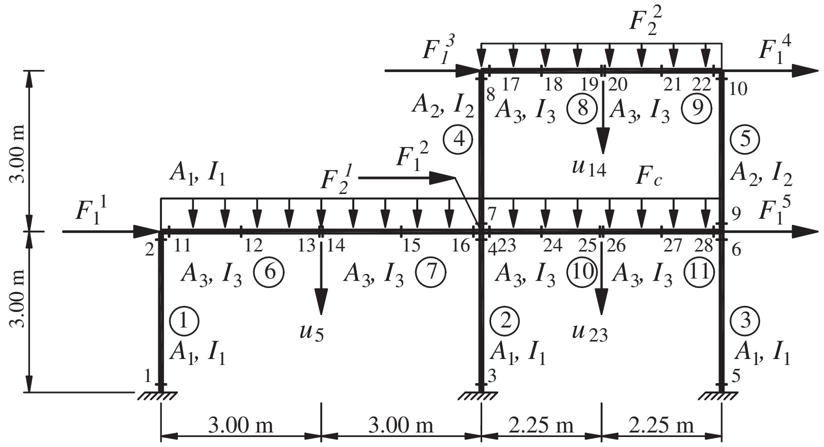


Fig. 4. Discretized frame.

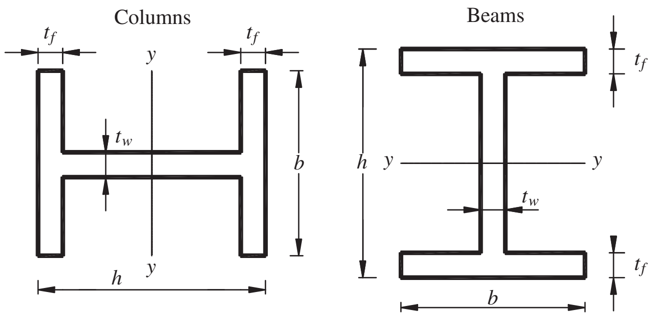


Fig. 5. Cross-sectional shapes for frame columns and beams.

lower bound total displacements constraints $\boldsymbol{u}_{min} = -\infty$ (displacements aren't limited).

4.2. Example 1

The plastic moments minimization problem (11)–(15) with stability constraints calculated according to the EC3 standard is investigated in this example. The limits of load variations are $\boldsymbol{F}_{1,inf} = \{-9.75, -4.9, -5, -6.75, -19.5\}$ kN, $\boldsymbol{F}_{1,sup} = \{13, 6.5, 6.75, 5, 14.6\}$ kN, $\boldsymbol{F}_{2,inf} = \{0, 0\}$, $\boldsymbol{F}_{2,sup} = \{48, 48\}$ kN/m. The parameters b and h remain the same throughout the optimization process, with only the thickness $t = t_f = t_w$ of the flanges and web varying. The values b and h of the cross-sections are given in Table. 5. In the case of discrete optimization, the cross-sections are selected from an assortment of available manufactured cross-sections.

The limit forces for the cross-sections when $t = t_f = t_w$ are calculated according to $M_0 = \sigma_y W_{pl,y} = \sigma_y\left(t^3 - (b+h)t^2 + \left(\frac{h^2}{4} + bh\right)t\right)$, $N_0 = \sigma_y A = \sigma_y(2bt + t(h-2t))$.

The main task is to determine the minimal plastic moments of the affected frame (Fig. 4). The frame plastic moments minimization is performed using mathematical model (11)–(15). The unknowns are plastic moments $\boldsymbol{M}_0$, and the vector of plasticity multipliers, λ_j, $j = 1, 2, \ldots, 4$. Five calculation cases were investigated:

Case C1. Only strength constraints (12) are taken into account. Optimization is continuous.

Case C2. Only strength (12) and stiffness (15) constraints are evaluated. The following total displacement constraints are imposed: $-\infty \leqslant u_5 \leqslant 0.03$ m, $-\infty \leqslant u_{14} \leqslant 0.0225$ m, $-\infty \leqslant u_{23} \leqslant 0.0225$ m (Fig. 4). Optimization is continuous.

Case C3. Only strength (12) and structural constraints (14) are taken into account. Optimization is continuous.

Case C4. Only strength (12) and structural constraints (14) are taken into account. Optimization is discrete.

Case C5. All constraints (strength (12), stiffness (15), and structural (stability) (14)) are evaluated. The following total displacement constraints are imposed: $-\infty \leqslant u_5 \leqslant 0.03$ m, $-\infty \leqslant u_{14} \leqslant 0.0225$ m, $-\infty \leqslant u_{23} \leqslant 0.0225$ m (Fig. 4). Optimization is continuous.

The calculation cases C1 and C2 was solved using the software M0opt1, whereas for the cases C3–C5 the software coupling M0opt1 – MatrixFrame, using the sequence of operations described in Section 2 and Fig. 2 was used.

The calculated results for all the cases described within the imposed constraints are shown in Table. 6. In cases C2 and C5, the total displacement u_{23} reaches the upper bound $u_{max} = 0.0225$ m. When discrete optimization is used in case C4, the limit moments $M_{01} = 174986$ N m, $M_{02} = 57610$ N m, and $M_{03} = 189018$ N m correspond to cross-sections HE240, HE160, and IPE330, respectively. It is noteworthy, that the same discrete cross-sections were obtained in 4th and 5th iterations, and therefore the optimization process was stopped and assumed that optimal solution was reached. The discrete optimization (case C4) is very important for civil engineering, however the continuous optimization (cases C1–C3, C5) could be an introductory step to discrete optimisation. For example, using section properties, obtained from the continuous optimization, is possible to choose nearest fitting discrete cross-section from assortment.

Convergence of the main optimization-problem objective function within the desired accuracy is a criterion of the optimal solution. In case C2, with a convergence tolerance $\delta = 0.25\%$, the iteration process is shown in Table 7. Convergence of the optimization-problem objective function for all cases is illustrated in Fig. 6.

Table 5
Values of cross-sections.

Elements $k = 1, 2, \ldots, s$	b (m)	h (m)
1, 2, 3	0.15	0.15
4, 5	0.1	0.12
6, 7, 8, 9, 10, 11	0.15	0.2

Table 6
Calculated results for the volume-minimization problem.

Case	M_{01} (N m)	M_{02} (N m)	M_{03} (N m)	Objective function (OF)	Volume (m^3)	Location of the plastic strains
C1	75441	41673	204168	3991522	0.26149777	6, 2, 23
C2	93970	34942	223206	4403462	0.292369813	23
C3	120537	48302	186579	4173339	0.283231289	23
C4	174986	57610	189018	4755802	0.350856685	23
C5	108090	44151	215258	4466587	0.300776204	23

Table 7
Convergence of the optimization-problem objective function for case C2.

Iteration	M_{01} (N m)	M_{02} (N m)	M_{03} (N m)	OF	δ OF%
1	96888	42400	240460	4733292	
2	93807	37591	204883	4143051	12,47
3	95221	37257	236064	4621487	−11, 55
4	93755	35439	211158	4223807	8, 61
5	94299	35814	231966	4543060	−7, 56
6	93670	34931	215459	4284503	5, 69
7	94140	35320	228876	4492323	−4, 85
8	93767	34832	218090	4324254	3, 74
9	94083	35129	226802	4459547	−3, 13
10	93840	34837	219776	4350228	2, 45
11	94044	35043	225444	4438312	−2, 02
12	93885	34860	220870	4367176	1, 60
13	94016	34999	224559	4424527	−1, 31
14	93912	34882	221583	4378244	1, 05
15	93997	34973	223983	4415558	−0, 85
16	93929	34898	222047	4385447	0, 68
17	93984	34958	223609	4409735	−0, 55
18	93939	34909	222348	4390121	0, 44
19	93975	34948	223365	4405942	−0, 36
20	93946	34916	222545	4393195	0, 29
21	93970	34942	223206	4403462	−0, 23

4.3. Example 2

The load-optimization problem (16)–(20) with stability constraints calculated according to the NEN 6771 standard is analyzed in this example.

The values of the cross-sections are shown in Table 8. The cross-sections remain unchanged throughout the entire optimization process. Limits for load variations $\boldsymbol{F}_{1,\inf} \leqslant \boldsymbol{F}_1 \leqslant \boldsymbol{F}_{1,\sup}$, $\boldsymbol{F}_{2,\inf} \leqslant \boldsymbol{F}_2 \leqslant \boldsymbol{F}_{2,\sup}$ are unknowns of the optimization problem. The loads $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ represents the wind and snow loads, respectively. The snow load can't act from bottom to top, so the constraint $\boldsymbol{F}_{2,\min} = -10 \leqslant \boldsymbol{F}_{2,\inf} \leqslant 0$ was applied for load $\boldsymbol{F}_2$ variation bound $\boldsymbol{F}_{2,\inf}$. Predicted optimal value is in range of ten to hundred thousands and it is possible to treat $\boldsymbol{F}_{2,\min} = -10 \cong 0$. The main task is to determine the load-variation bounds of the affected frame (Fig. 4). The frame load optimization is performed using mathematical model (16)–(20). The unknowns are the load-variation bounds, $\boldsymbol{F}_{1,\inf}$, $\boldsymbol{F}_{2,\inf}$, $\boldsymbol{F}_{1,\sup}$, and $\boldsymbol{F}_{2,\sup}$, and the vector of plasticity multipliers, λ_j, $j = 1, 2, \ldots, 4$. Three calculation cases were investigated:

Case C1. Only strength constraints (17) are taken into account.

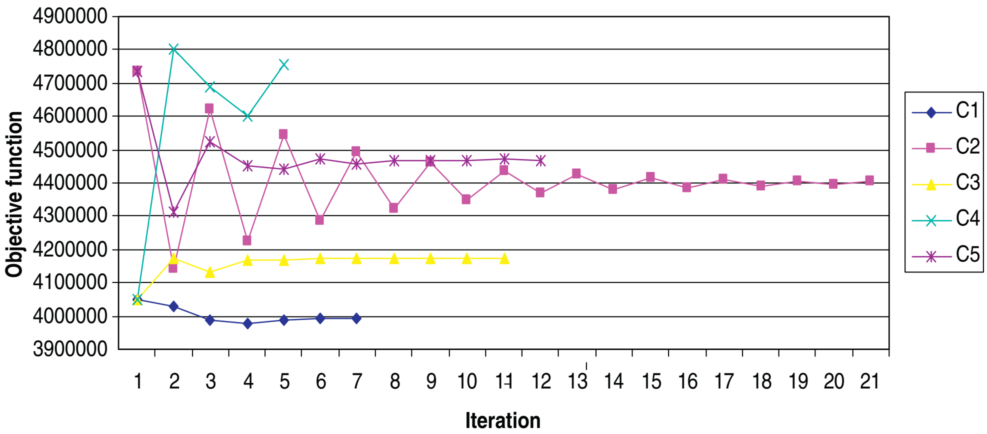


Fig. 6. Convergence of the optimization-problem objective function.

Table 8
Values of cross-sections.

Elements $k = 1, 2, \ldots, s$	b (m)	h (m)	t (m)	A_k (m^2)	M_{0k} (N m)	N_{0k} (N)
1, 2, 3	0.15	0.15	0.016	0.006688	88665	1571680
4, 5	0.1	0.12	0.01	0.003000	31725	705000
6, 7, 8, 9, 10, 11	0.15	0.2	0.03	0.013200	21432	3102000

Table 9
Calculated results for the load-optimization problem.

Case	$\boldsymbol{F}_{1,\sup}$ (N)	$\boldsymbol{F}_{2,\sup}$ (N/m)	$\boldsymbol{F}_{1,\inf}$ (N)	$\boldsymbol{F}_{2,\inf}$ (N/m)	OF	Location of the plastic strains
C1	23679	44035	−29349	−10	97073	4, 6, 8, 23
C2	15777	26006	−23958	−10	65751	4, 6
C3	11839	19200	−14673	−10	45722	4

Case C2. Strength (17) and stiffness (20) constraints are taken into account. The following total displacement constraints are imposed: $-\infty \leqslant u_5 \leqslant 0.03$ m, $-\infty \leqslant u_{14} \leqslant 0.0225$ m, $-\infty \leqslant u_{23} \leqslant 0.0225$ m (Fig. 4).

Case C3. Strength (17) and structural constraints (19) are taken into account.

The calculation cases C1 and C2 was solved using the software MaxFopt1, whereas for the cases C3 the software coupling M0opt1–MatrixFrame, using the sequence of operations described in Section 2 and Fig. 3, was used.

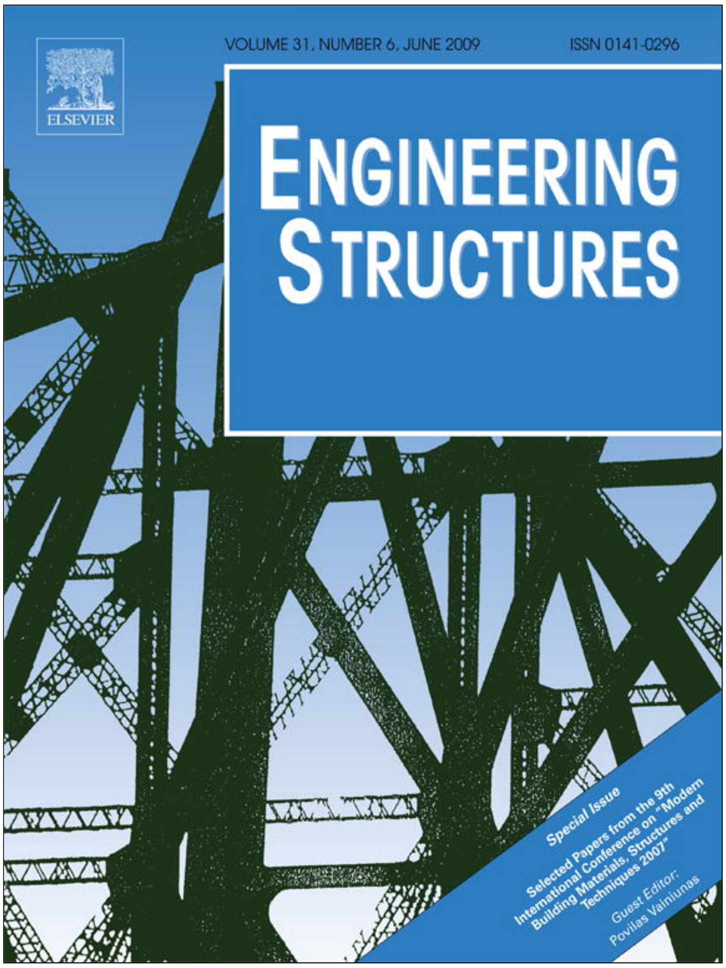
The calculated results for all cases described within the imposed constraints are presented in Table 9. In case C2, the total displacement u_{23} reaches the upper bound $u_{max} = 0.0225$ m. In presented example the stability evaluation plays important role. In case C3 the value of objective function (OF) is the smallest. The difference of OF value between C3 and C2 is 44% and between C3 and C1 is 112%. The iterative solution procedure was performed only for case C3, while the optimal solutions for cases C1 and C2 were obtained in the first iteration. Only one iteration was needed because no software coupling was used and the stiffness matrix $\boldsymbol{K}$ is constant in the whole optimization process.

5. Conclusion

Practical implementation of a shakedown structural-design methodology should be based, not only on theoretical improvements and new mathematical models, but also on a close relation with existing building design practices. In this way, it is possible to avoid a gap between the theoretical methods of structural optimization and real design practices based on standards. For this purpose, this paper presents main optimization problems with strength, stiffness, and stability constraints, in which the part of the solution related to stability is transferred to a design software package which conforms to implemented standards. The solution procedure therefore becomes iterative: the structural or load constraints for an ordinary iteration of the main optimization problem are calculated using the design software. On the other hand, the initial data for the design software become residual forces and residual displacements obtained from the solution of the optimization problem, i.e., the influence of plastic deformations is evaluated. Convergence of the main optimization-problem objective function to the desired degree of accuracy is a criterion of the optimal solution. The proposed ways of solving optimization problems include the implementation of discrete-optimization principles. For future investigators, the methodology developed here offers the possibility of integrating the solution of nonlinear programming problems (plastic state variables – residual forces and displacements) into their structural design software. In this way, shakedown theory can become a generalized tool for calculation and optimization of elastic–plastic structures under different loading conditions.

References

[1] Atkočiūnas J, Venskus A. Optimal shakedown design of frames under stability conditions. In: Topping BHV, Papadrakakis M, editors. Proceedings of the ninth international conference on computational structures technology. Stirlingshire, United Kingdom: Civil-Comp Press; 2008. Paper 159.
[2] Kaneko L, Maier G. Optimum design of plastic structures under displacement's constraints. Comput Methods Appl Mech Eng 1981;27(3):369–92.
[3] Stein E, Zhang G, Mahnken R. Shakedown analysis for perfectly plastic and kinematic hardening materials. In: CISM. Progress in computernal analysis or inelastic structures. Wien, New York: Springer-Verlag; 1993. p. 175–244.
[4] Giambanco F, Palizzolo L, Polizzotto C. Optimal shakedown design of beam structures. Struct Optim 1994;8:156–67.
[5] Tin-Loi F. Optimum shakedown design under residual displacement constraints. Struct Multidiscip Optim 2000;19(2):130–9.
[6] Kaliszky S, Lógó J. Plastic behaviour and stability constraints in the shakedown analysis and optimal design of trusses. Struct Multidiscip Optim 2002;24(2):118–24.
[7] Choi SH, Kim SE. Optimal design of steel frame using practical nonlinear inelastic analysis. Eng Struct 2002;24(9):1189–201.
[8] Staat M, Heitzer M, editors. Numerical methods for limit and shakedown analysis. Series of John von Neumann institute for computing, vol. 15. 2003. p. 306.
[9] Kojic M, Bathe K-J. Inelastic analysis of solids and structures, vol. 414. New York: Springer; 2005.
[10] Benfratello S, Cirone L, Giambanco F. A multicriterion design of steel frames with shakedown constraints. Comput Struct 2006;84:269–82.
[11] Borkowski A, Atkočiūnas J. Optimal design for cyclic loading. In: IUTAM, Symposium on optimization in structural design. Held in Warsaw on August 21–24, 1973. Springer-Verlag; 1975.
[12] Atkočiūnas J. Mathematical models of optimization problems at shakedown. Mech Res Commun 1999;26(3):319–26.
[13] Cyras AA. Mathematical models for the analysis and optimization of elastoplastic structures. Chichester: Ellis Horwood Lim; 1983. p. 121.
[14] Atkočiūnas J, Borkowski A, König JA. Improved bounds for displacements at shakedown. Comput Methods Appl Mech Eng 1981;28(3):365–76.
[15] Dorosz S, König JA. An iterative method of evaluation of elastic–plastic deflections of hyperstatic framed structures. Ing Arch 1985;55:202–12.
[16] Maier G, Comi C, Corigliano A, Perego U, Hübel H. Bounds and estimates on inelastic deformations: a study of their practical usefulness. European Commission Report. Nuclear Science and Technology Series, vol. 286. Brussels: European Commission; 1996.
[17] Hachemi A, Weichert D. Application of shakedown theory to damaging inelastic material under mechanical and thermal loads. Int J Mech Sci 1997;39(9):1067–76.
[18] Lange-Hansen P. Comparative study of upper bound methods for the calculation of residual deformation after shakedown. Series R, 49. Lyngby: Technical Department of Structural Engineering and Materials, University of Denmark; 1998. p. 74.
[19] Merkevičiūtė D, Atkočiūnas J. Optimal shakedown design of metal structures under stiffness and stability constraints. J Constr Steel Res 2006;62(12): 1270–5.
[20] Atkočiūnas J, Merkevičiūtė D, Venskus A. Optimal shakedown design of bar systems: strength stiffness and stability constraints. Comput Struct 2008;86:1757–68.
[21] Bazaraa MS, Sherali HD, Shetty CM. Nonlinear programming: theory and algorithms, vol. 652. New York: Brijbasi Art Press Ltd., John Wiley and Sons, Inc.; 2004.


ELSEVIER
VOLUME 31, NUMBER 6, JUNE 2009
ISSN 0141-0296
ENGINEERING STRUCTURES
Special Issue
Selected Papers from the 9th International Conference on "Modern Building Materials, Structures and Techniques 2007"
Guest Editor: Povilas Vainiunas

Engineering Structures 31 (2009) 1298–1304

Contents lists available at ScienceDirect

Engineering Structures

journal homepage: www.elsevier.com/locate/engstruct

Discrete optimization problems of the steel bar structures

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A R T I C L E I N F O

Article history:
Available online 1 February 2009

Keywords:
Elastic and elastic–plastic steel structures
Discrete optimization
Finite element method
Mathematical programming

A B S T R A C T

In this paper there are considered the optimal design problems of the elastic and elastic–plastic bar structures. These problems are formulated as nonlinear discrete optimization problems. The cross-sections of the bars are designed from steel rolled profiles. The mathematical models of the problems, including the structural requirements of strength, stiffness and stability, are formulated in terms of the finite element method. The stated nonlinear optimization problems are solved by the iterative method, where each iteration comprises of the selection of the cross-sections of the bars from the assortment and solution of the linear problems of discrete programming. The requirement of discrete cross-sections is ensured by the branch and bound method.

1. Introduction

For the purpose of saving material, structures are designed by applying the methods of optimization [1–7]. The various specific algorithms for nonlinear optimization problems of structures are recently created: incremental [8], genetic [9–11], discrete optimization [5], evolutionary [12], homogenization [13] and other optimization algorithms [14–16]. The solution algorithms for nonlinear optimization problems are not as universal as the latter for the linear problems. They are mostly dedicated to solving a particular type of problem. Furthermore, the problem of convergence of finding an optimal solution occurs frequently, while they are applying. Therefore, nonlinear optimization problems frequently are solved by using the approximation technique when the linear programming problem is solved in each iteration. This method is applied in the paper [17], which is dedicated to the optimization of elastic structures. While designing the structures, an additional economy of the structural material is obtained for the structures with plastic deformations with respect to optimal ones with elastic deformations. However, the optimization problems of elastic–plastic structures [6–9,15] are evaluated where not only the strength, but also stiffness and stability requirements, are complex nonlinear programming problems and the realization of them is complicated. In this paper design problems of the elastic and elastic–plastic steel structures are investigated. Their mathematical models are formulated as nonlinear mathematical programming problems by taking into account requirements of design codes. Mathematical models are created by using the finite element method. In these models there are evaluated the conditions of strength, stiffness and stability [18]. The cross-sections are designed from standard steel rolled profiles. The formulated nonlinear optimization problems are solved by the iterative method where each iteration comprises the selection of the cross-sections of the bars from the assortment and solution of the linear problems of discrete programming. The requirement of discrete cross-sections is ensured by the branch and bound method.

2. The volume minimization problem for elastic structures

2.1. Mathematical models

There is considered the bar structure loaded by load combinations $v = 1, 2, \ldots, p$, which bars are designed from steel rolled profiles set Π. Let the vector $\mathbf{A}_0$ denote the structural bars' cross-sectional areas and $\mathbf{F}_v$, $\mathbf{S}_v$, $\mathbf{u}_v$ define the load, internal forces and displacements of v-th load combination, respectively. Then the volume (mass) minimization problem for the elastic structure is expressed by the following mathematical model:

$$\begin{aligned}
&\text{find} && \min f = \mathbf{L}^{\mathrm{T}}\mathbf{A}_0 \\
&\text{subject to} && [A]\,\mathbf{S}_v = \mathbf{F}_v, \quad \left[\bar{D}\right]\mathbf{S}_v - [A]^{\mathrm{T}}\mathbf{u}_v = \mathbf{0}, \\
& && [G]\,\mathbf{A}_0 - \left[\bar{\Phi}\right]\mathbf{S}_v \geq \mathbf{0}, \quad [E]\,\mathbf{u}_v \leq \mathbf{u}^{+}, \\
& && v = 1, 2, \ldots, p; \quad \mathbf{A}_0 \geq \mathbf{A}_0^{-}, \quad \mathbf{A}_0 \in \Pi.
\end{aligned} \tag{1}$$

In this model: equalities—equilibrium and geometrical equations, describing the structural forces and displacements; first inequality— strength and stability conditions; other inequalities—displacements (stiffness) and constructive constraints. $\mathbf{L}$ is the vector of the structural elements' lengths. The unknowns of this problem are the vectors $\mathbf{A}_0$, $\mathbf{S}_v$ and $\mathbf{u}_v$. Thus, the objective function of the problem expresses volume and the mass of the structure at the same time. Flexibility matrix $\left[\bar{D}\right]$ of the structural elements

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0141-0296/$ – see front matter © 2009 Elsevier Ltd. All rights reserved.
doi:10.1016/j.engstruct.2009.01.004

together with the strength and stability matrix $[\bar{\Phi}]$ depend on the unknown $\mathbf{A}_0$. Therefore the model (1) is the nonlinear programming problem: the cross-sections of the structural bars, satisfying the requirements of the minimum volume (mass) of the structure, strength, stiffness and stability, are searched for.

By eliminating the internal forces $\mathbf{S}_v = [\bar{D}]^{-1}[A]^{\mathrm{T}}\mathbf{u}_v$ and geometrical equations, this model can be rewritten as the following optimization problem:

$$\begin{aligned} &\text{find} && \min f = \mathbf{L}^{\mathrm{T}}\mathbf{A}_0 \\ &\text{subject to} && [\bar{K}]\,\mathbf{u}_v = \mathbf{F}_v, \quad [G]\mathbf{A}_0 - [\bar{\Phi}_u]\,\mathbf{u}_v \ge \mathbf{0}, \\ & && [E]\,\mathbf{u}_v \le \mathbf{u}^+, \quad v = 1, 2, \ldots, p; \qquad \mathbf{A}_0 \ge \mathbf{A}_0^-, \\ & && \mathbf{A}_0 \in \Pi. \end{aligned} \tag{2}$$

where $[\bar{\Phi}_u] = [\bar{\Phi}]\,[D]^{-1}[A]^{\mathrm{T}}$; $[\bar{K}] = [A]\,[\bar{D}]^{-1}[A]^{\mathrm{T}}$ is the global stiffness matrix of the structure.

2.2. Formulation of the main dependencies

The main dependencies composing the problems (1) and (3) are formulated in terms of the finite element method. For this purpose the structure is divided into the elements (bars) $k = 1, 2, \ldots, r$ joined in the nodes. The dependencies of the model (1) can be composed by using the equilibrium finite element method [19], and the model (3) can be created with the help of the equilibrium or geometrically compatible finite element method [20], because the stiffness matrix $[\bar{K}]$ can be formulated not only from the indicated formula, but also from the stiffness matrices of elements too.

Two equation groups compose *the equilibrium equations* $[A]\,\mathbf{S}_v = \mathbf{F}_v$:

(1) *the equilibrium equations for nodes* describing the relation between the nodal forces of connected into nodes elements and the external forces acting on the nodes;
(2) *the equilibrium equations for elements* describing the relation between the nodal forces and acting on the element external load, and are formulated only for elements affected by a distributed load. Expressions of these equations are presented in the papers [17,19].

The equilibrium equation matrix $[A]$ could be formulated from the coefficients of the equilibrium equations of nodes and elements or from the formula $[A] = [C]^{\mathrm{T}}[\bar{A}]$ [19]; here the compatibility matrix $[C]$ describing the relation between global displacements of the structural nodes and nodal displacements of elements; $[\bar{A}] = \mathrm{diag}\,[A_k]$ is the quasi-diagonal matrix, whose diagonal submatrices are composed from the coefficients of the static equations $\mathbf{P}_k = [A_k]\,\mathbf{S}_k$ of the elements.

Flexibility matrix $[\bar{D}] = \mathrm{diag}\,[D_k]$ of *geometrical equations* $[\bar{D}]\,\mathbf{S}_v - [A]^{\mathrm{T}}\mathbf{u}_v = 0$ contains in the principal diagonal the flexibility matrices of the finite elements $[D_k]$. Its coefficients are calculated by formula $d_{ij} = d_k \int_{l_k} H_{ki}(x)\,H_{kj}(x)\,\mathrm{d}x$, where $H_{ki}(x)$ is the shape function of the internal forces; flexibility of the element under tension or compression is $d_k = 1/EA_k$, flexibility of an element under bending is $d_k = 1/EI_k$; E is the elasticity modulus, A_k, I_k are the cross-sectional area and moment of inertia, respectively.

First-order and second-order approximation functions of forces (bending moments and axial forces) for equilibrium finite elements and expressions of flexibility matrix $[D_k]$ and equilibrium equations are presented below.

(a) Expressions of first-order element (Fig. 1):

$$M_k(x) = \sum_{j=1}^{2} H_{kj}(x)\,M_{kj} = \left(1 - \frac{x}{l_k}\right)M_{k1} + \frac{x}{l_k}M_{k2},$$

$$N_k(x) = N_k;$$

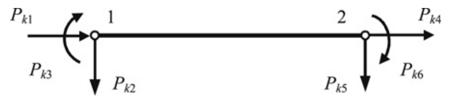


Fig. 1. First-order element.

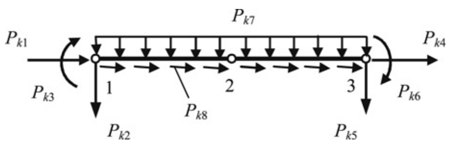


Fig. 2. Second-order element.

$$\mathbf{P}_k = \begin{bmatrix} P_{k1} \\ P_{k2} \\ P_{k3} \\ P_{k4} \\ P_{k5} \\ P_{k6} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1/l_k & -1/l_k & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \\ -1/l_k & 1/l_k & 0 \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} M_{k1} \\ M_{k2} \\ N_k \end{bmatrix} = [A_k]\,\mathbf{S}_k,$$

$$[D_k] = \frac{l_k}{6EI_k}\begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 6I_k/A_k \end{bmatrix};$$

(b) Expressions of second-order element (Fig. 2) subjected to distributed load:

$$\begin{aligned} M_k(x) &= \sum_{j=1}^{3} H_{kj}(x)\,M_{kj} = \left(1 - \frac{3x}{l_k} + \frac{2x^2}{l_k^2}\right)M_{k1} \\ &\quad + \left(\frac{4x}{l_k} - \frac{4x^2}{l_k^2}\right)M_{k2} + \left(-\frac{x}{l_k} + \frac{2x^2}{l_k^2}\right)M_{k3}, \\ N_k(x) &= \left(1 - \frac{x}{l_k}\right)N_{k1} + \frac{x}{l_k}N_{k3}; \end{aligned}$$

$$\mathbf{P}_k = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 3/l_k & -4/l_k & 1/l_k & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1/l_k & -4/l_k & 3/l_k & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ -4/l_k^2 & 8/l_k^2 & -4/l_k^2 & 0 & 0 \\ 0 & 0 & 0 & 1/l_k & -1/l_k \end{bmatrix} \cdot \begin{bmatrix} M_{k1} \\ M_{k2} \\ M_{k3} \\ N_{k1} \\ N_{k3} \end{bmatrix},$$

$$[D_k] = \frac{l_k}{15EI_k}\begin{bmatrix} 2 & 1 & -0,5 & 0 & 0 \\ 1 & 8 & 1 & 0 & 0 \\ -0,5 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 5i_k & 2,5i_k \\ 0 & 0 & 0 & 2,5i_k & 5i_k \end{bmatrix},$$

where $i_k = I_k/A_k$.

The matrices $[A_k]$ and $[D_k]$ for elements under tension or under bending can be obtained by removing corresponding columns and rows.

Strength and stability condition. Strength condition of the element under bending and tension or compression of the j-th section is described via inequalities:

$$\begin{aligned} &N_j + c_jM_j - RA_j \le 0, \qquad -N_j + c_jM_j - RA_j \le 0, \\ &N_j - c_jM_j - RA_j \le 0, \qquad -N_j - c_jM_j - RA_j \le 0. \end{aligned} \tag{3}$$

Here $R = f_{y,d}\gamma_c$; $f_{y,d}$ is the yield strength; γ_c is the partial factor of the exploitation conditions; $c_j = A_j/W_{ej}$; A_j, W_{ej} are the cross-sectional area and section modulus, respectively.

Furthermore, the bars under compression must satisfy the stability condition

$$-N_j/\varphi_j \le RA_j \quad \text{or} \quad -N_j/\varphi_j - RA_j \le 0. \tag{4}$$

Strength (3) and stability (4) conditions of elements meet the Lithuanian national standards of civil engineering [18]. However, in the general case the conditions of strength and stability of elements can be formulated according to other design codes, for example Eurocode 3 [21]. Strength conditions (3) are created for all nodes of elements and stability conditions (4) only for the elements under compression. All of them are described via inequality $[G]\ \mathbf{A}_0 - \left[\overline{\Phi}\right]\ \mathbf{S}_\upsilon \geq \mathbf{0}$.

2.3. Solution algorithms

The direct solution of the nonlinear discrete programming problems (1) and (3) is fairly complicated. However, their solutions can be found in the iterative process, where in each iteration the cross-sectional profile is selected from the assortment and the linear programming problem solutions, which are obtained when matrices $\left[\bar{D}\right], \left[\overline{\Phi}\right]$ and $\left[\bar{K}\right], \left[\overline{\Phi}_u\right]$ of models (1) and (3) are replaced by matrices $[D]$, $[\Phi]$ and $[K]$, $[\Phi_u]$, in which all coefficients are known, because the cross-sections of bars are set. The iterative process is finished, when it is found cross-sectional areas coincide with the previously set ones. For the purpose of minimizing problem volume it is possible to consider each load case separately and for every one solve such a problem:

$$\begin{aligned} &\text{find} && \min f = \mathbf{L}^{\mathrm{T}}\mathbf{A}_{0\upsilon} \\ &\text{subject to} && [A]\ \mathbf{S}_\upsilon = \mathbf{F}_\upsilon, \quad [D]\ \mathbf{S}_\upsilon - [A]^{\mathrm{T}}\ \mathbf{u}_\upsilon = \mathbf{0}; \\ &&& [G]\ \mathbf{A}_{0\upsilon} - [\Phi]\ \mathbf{S}_\upsilon \geq \mathbf{0}, \quad [E]\mathbf{u}_\upsilon \leq \mathbf{u}^+; \\ &&& \mathbf{A}_{0\upsilon} \geq \mathbf{A}_{0,\upsilon-1}, \quad \mathbf{A}_{0\upsilon} \in \Pi \end{aligned} \tag{5}$$

or

$$\begin{aligned} &\text{find} && \min f = \mathbf{L}^{\mathrm{T}}\mathbf{A}_{0\upsilon} \\ &\text{subject to} && [K]\ \mathbf{u}_\upsilon = \mathbf{F}_\upsilon; \\ &&& [G]\ \mathbf{A}_{0\upsilon} - [\Phi_u]\ \mathbf{u}_\upsilon \geq \mathbf{0}, \quad [E]\ \mathbf{u}_\upsilon \leq \mathbf{u}^+; \\ &&& \mathbf{A}_{0\upsilon} \geq \mathbf{A}_{0,\upsilon-1}, \quad \mathbf{A}_{0\upsilon} \in \Pi. \end{aligned} \tag{6}$$

Inequality $\mathbf{A}_{0\upsilon} \geq \mathbf{A}_0^-$ for the load cases $\upsilon > 1$ is replaced by the condition $\mathbf{A}_{0\upsilon} \geq \mathbf{A}_{0,\upsilon-1}$. The vector $\mathbf{A}_{0p}$ corresponding to the last load case is the solution of the problems (1) and (3).

Furthermore, the optimization problems (5) and (6) can be solved in two stages:

(1) classic problem of structural mechanics is solved i.e. the displacements $\mathbf{u}_\upsilon = [K]^{-1}\,\mathbf{F}_\upsilon$ and internal forces $\mathbf{S}_\upsilon = [D]^{-1}\,[A]^{\mathrm{T}}\,\mathbf{u}_\upsilon$ are calculated; for this can be applied the equilibrium or geometrically compatible finite element method and various state-of-the-art computer technologies dedicated for this kind of problems;

(2) it is determining the vector of strength and stability conditions $\mathbf{S}_{0\upsilon} = [\Phi]\ \mathbf{S}_\upsilon$ and solving the minimization problem:

$$\begin{aligned} &\text{find} && \min f = \mathbf{L}^{\mathrm{T}}\mathbf{A}_0 \\ &\text{subject to} && [G]\ \mathbf{A}_0 \geq \mathbf{S}_{0\upsilon}, \quad [G_0]\ \mathbf{A}_0 \geq [E]\ u_\upsilon, \\ &&& \mathbf{A}_0 \geq \mathbf{A}_0^-, \quad \mathbf{A}_0 \in \Pi, \quad \upsilon = 1, 2, \ldots, p. \end{aligned} \tag{7}$$

Here the vector $\mathbf{A}_0$ is unknown, whereas $\mathbf{S}_{0\upsilon} = [\Phi]\ \mathbf{S}_\upsilon$. Having software for the internal forces calculations, the solution method is easier, because the volume of this problem is smaller. It should be noted that it is possible to search for the optimal solution when stability requirements are neglected. But in this case it is necessary to verify if received cross-sections of bars under compression satisfy stability conditions. If they are violated, then cross-sections should be augmented and additional calculation iterations should be performed and included into the mathematical model stability conditions.

In the following optimization problems, the value of reduction factor φ for eccentrically compressed elements is determined by national standards of civil engineering [18] by taking into account the eccentricity of the compression force, the slenderness of the element and form coefficient of cross-sectional shape. In each iteration value of eccentricity is determined by internal forces obtained in previous iteration and values c_j of strength conditions are determined by choosing characteristics of cross-sectional shape A_j and $W_{ej}(W_{pj})$.

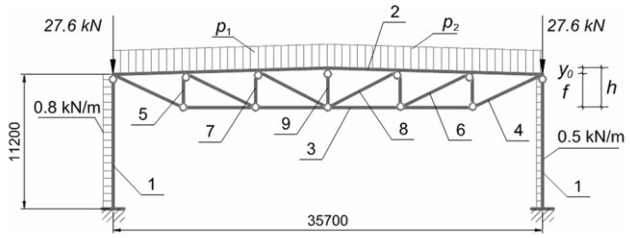


Fig. 3. Calculation schema of the framed truss.

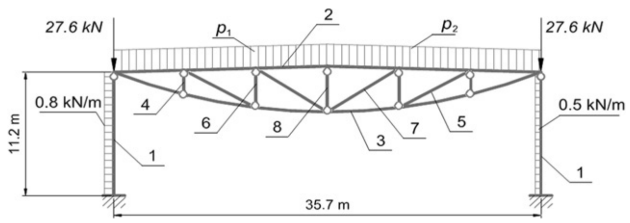


Fig. 4. Framed truss with parabolic sketch bottom chord.

Example 1. Let the bar structure, shown in Fig. 3 be loaded by three load cases: I – $p_1 = 16.4$ kN/m, $p_2 = 16.4$ kN/m; II – $p_1 = 16.4$ kN/m, $p_2 = 4$ kN/m; III – $p_1 = 4$ kN/m, $p_2 = 16.4$ kN/m. Moreover, the vertical load $F = 27.6$ kN and indicated wind load acts in each load case. The optimal cross-sections from steel rolled profiles must be found. Columns and the upper chord are designed from I profiles and other bars from hollow rectangle tubes. Yield strength $R_y = 275$ MPa, elasticity module $E = 2.1 \times 10^5$ MPa. Stiffness requirements are described via constraints $u_x \leq 5$ cm and $u_y \leq 10$ cm, where u_x is the horizontal displacement of top node of the column; u_y is the vertical displacement in the middle of the bottom chord of the truss.

The columns and the upper chord are calculated as the elements under bending and compression and the other ones are calculated as the elements under tension or compression. Cross-sections are selected from the assortment. Initial height of the truss $h = 3.3$ m. After optimization the following cross-sections were obtained: 1 – HEA300; 2 – IPE330; 3–180 × 180 × 6; 4–150 × 150 × 5; 5–90 × 90 × 5; 6–90 × 90 × 4; 7–70 × 70 × 4; 8–80 × 80 × 4; 9–60 × 60 × 5. Total weight of the optimal structure is 5229 kg.

Optimization of the structure is influenced not only by the height of the truss, but also by the web shape and the length of the segments. For this purpose the problems of truss height and web shape were created and considered.

3. Truss height and web shape optimization problems

In this section there are considered and formulated the optimal height and the rational shape of bottom chord of the framed truss, shown in Fig. 3, search problems. Two designed versions are considering: (1) truss with horizontal bottom chord (Fig. 3); (2) truss with parabolic bottom chord (Fig. 4). Height optimization problems of these trusses are described by such mathematical models of Box I: Here s_1 is number of bottom chord bars; s_t–number of web bars; f–camber of the truss; l_j–length of j-th bar, $a_{ji} = 4x_i\,(l - x_i)\,/l^2$, l–length of the span; y_{0j}–the sketch of the truss upper node j with respect to the support nodes. The vectors of internal forces, displacements $\mathbf{S}_\upsilon$, $\mathbf{u}_\upsilon$ and design parameters of the structure – cross sectional areas A_j and sketch of the truss f are the unknowns of these problems. There are nonlinear programming problems, which can be solved iteratively.

(a) truss with parabolic bottom chord
find $\min \mathbf{L}^{\mathrm{T}}\mathbf{A}_0$
subject to

$$[A(\mathbf{l})]\,\mathbf{S}_v = \mathbf{F}_v, \qquad [D(\mathbf{l},\mathbf{A}_0)]\,\mathbf{S}_v - [A(\mathbf{l})]^{\mathrm{T}}\,\mathbf{u}_v = \mathbf{0},$$

$$[G]\mathbf{A}_0 - [\Phi(\mathbf{A}_0)]\mathbf{S}_v \geq \mathbf{0}, \quad [E]\mathbf{u}_v \leq \mathbf{u}^+,$$

$$v = 1, 2, \ldots, p;$$

$$l_j = \left[l_{jx}^2 + \left(y_{j2} + y_{0j}\right)^2\right]^{1/2}, \quad j = 1, 2, \ldots, s_1;$$

$$l_j = [l_{jx}^2 + (y_{j2} + y_{0j})^2]^{\frac{1}{2}}, \quad j = 1, 2, \ldots, s_t;$$

$$y_{ji} - a_{ji}f = 0, \quad i = 1, 2; \quad \mathbf{A}_0 \geq \mathbf{A}_0^-, \quad \mathbf{A}_0 \in \Pi;$$

(b) truss with horizontal bottom chord
find $\min \mathbf{L}^{\mathrm{T}}\mathbf{A}_0$
subject to

$$[A(\mathbf{l})]\,\mathbf{S}_v = \mathbf{F}_v,$$

$$[D(\mathbf{l},\mathbf{A}_0)]\mathbf{S}_v - [A(\mathbf{l})]^{\mathrm{T}}\mathbf{u}_v = \mathbf{0},$$

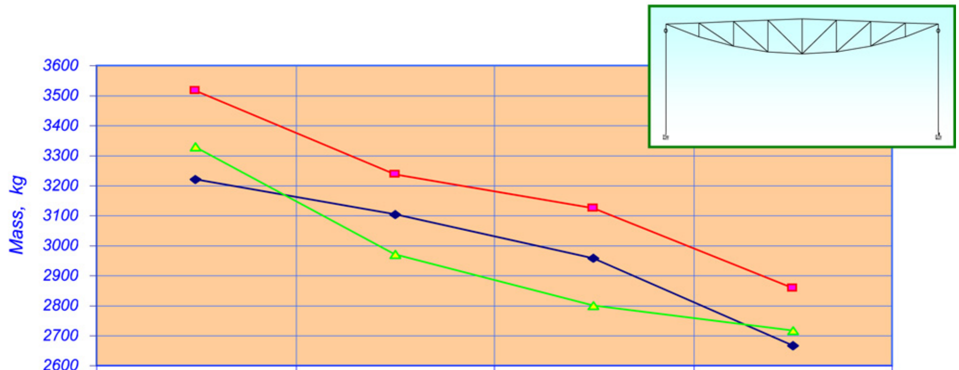
$$[G]\,\mathbf{A}_0 - [\Phi\,(\mathbf{A}_0)]\,\mathbf{S}_v \geq \mathbf{0},$$

$$[E]\,\mathbf{u}_v \leq \mathbf{u}^+, \qquad \mathbf{A}_0 \geq \mathbf{A}_0^-, \qquad \mathbf{A}_0 \in \Pi,$$

$$l_j = [l_{jx}^2 + (f + y_{0j})^2]^{1/2},$$

$$j = 1, 2, \ldots, s_t; \; v = 1, 2, \ldots, p.$$

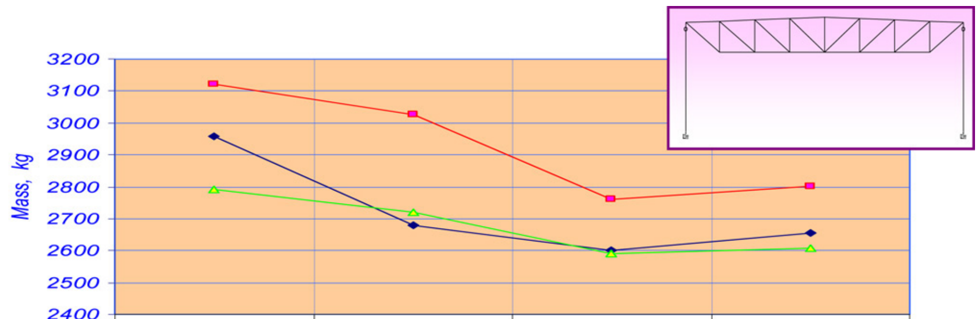
Box I.



	3.3	3.6	4.0	4.5
6 segm.	3519	3239	3123	2858
8 segm.	3219	3106	2956	2668
10 segm.	3330	2972	2800	2718

Height of the truss, m

Fig. 5. Mass of trusses with parabolic bottom chord dependence on height.



	3.3	3.6	4.0	4.5
6 segm.	3121	3027	2761	2801
8 segm.	2957	2679	2600	2657
10 segm.	2792	2721	2591	2606

Height of the truss, m

Fig. 6. Mass of N-shaped trusses with horizontal bottom chord dependence on height.

Example 2. For the analyses of the framed structure in the first example, which is loaded by three prescribed load cases, must be determined: (1) truss rational bottom chord sketch; (2) rational length of the web segment and bar placing; (3) optimal height of the truss. The investigations were performed for three types of trusses:

(1) N-shaped truss with parabolic bottom chord (Fig. 4);
(2) N-shaped truss with horizontal bottom chord (Fig. 3);
(3) M-shaped truss with horizontal bottom chord (Fig. 9).

The purpose of the investigation is the determination of the optimal height and the optimal segment count by comparing steel input and determination of minimal mass – economic truss. We investigated trusses of height $h = 3.3 \div 4.5$ m composed of 6, 8 and 10 segments. The results of frame optimal design are presented in Figs. 5–10. They show various truss mass dependencies on their count of segments and height. The results of the optimal design of N-shaped trusses with parabolic and horizontal bottom chord are presented in Figs. 5 and 6. They show that for any number of

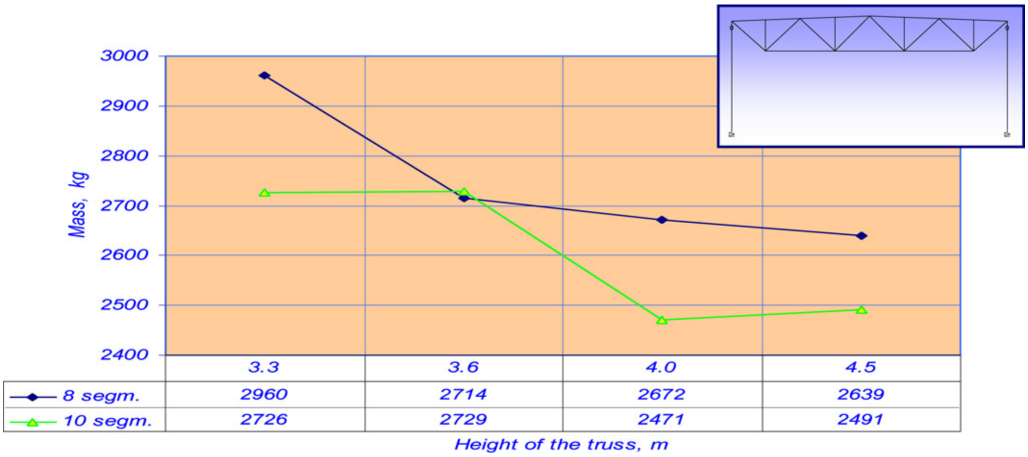


Fig. 7. Mass of M-shaped trusses with horizontal bottom chord dependence on height.

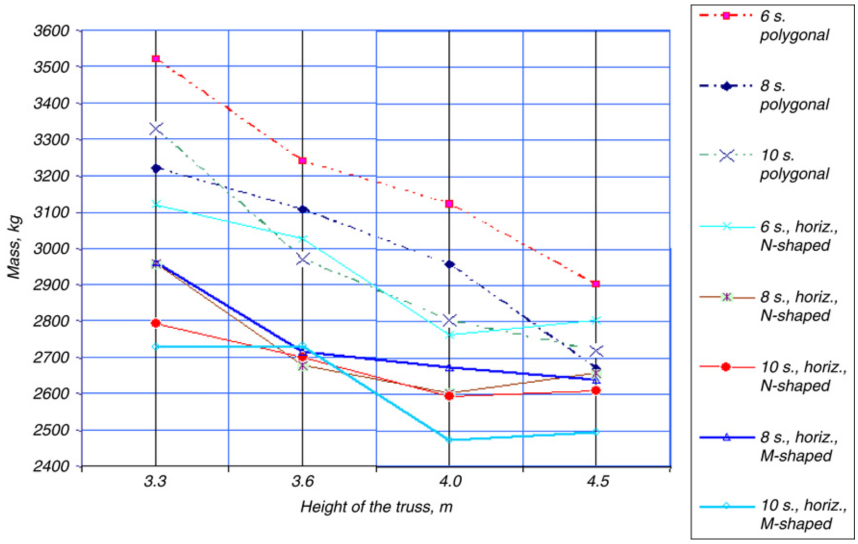


Fig. 8. Analysis results of various web and chord shapes.

Table 1
Mass of the truss.

Type of the truss	Number of the segments	Mass of the truss (kg)			
		$h = 3.3$ m	$h = 3.6$ m	$h = 4$ m	$h = 4.5$ m
1 (Fig. 4)	6	3519	3239	3123	2858
	8	3219	3106	2956	2668
	10	3330	2972	2800	2718
2 (Fig. 3)	6	3121	3027	2761	2801
	8	2957	2679	2600	2657
	10	2792	2721	2591	2606
3 (Fig. 9)	8	2960	2714	2672	2639
	10	2726	2729	2471	2491

segments the optimal height of the truss with parabolic bottom chord is $h = 4.5$ m, and latter of the truss with horizontal bottom chord – $h = 4$ m. The minimal mass of the first truss is $G = 2668$ kg (count of segments is $s = 8$), and latter of the second – $G = 2591$ kg. ($s = 10$) is less by 77 kg. In that case when count of segments and height are the same, the mass of truss with horizontal bottom chord in all cases is less. Therefore the truss with horizontal bottom chord is optimal (see Table 1).

Fig. 7 shows investigation results of an M-shaped truss with horizontal bottom chord and in Fig. 8 optimal design results of all three trusses are presented. By comparing the presented results we state that an M-shaped truss with horizontal bottom chord is

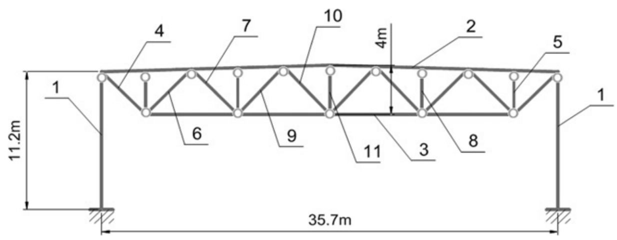


Fig. 9. Framed truss with the optimal shape web.

	3.3	3.4	3.5	3.6	3.7	3.8	3.9	4.0	4.1	4.2	4.3	4.4	4.5
Chords mass, kg	2117	2063	2063	2063	2009	1793	1793	1748	1748	1748	1711	1711	1702
Web mass, kg	609	628	639	665	677	700	712	723	740	752	764	777	789
Total mass, kg	2726	2691	2702	2729	2686	2493	2504	2471	2488	2500	2475	2487	2491

Fig. 10. Investigations results of the optimal web truss height.

optimal (Fig. 9) with count of segments $s = 10$, and height $h = 4$ m. Their mass is $G = 2471$ kg. Their graphical dependencies of the bottom chord, count of segments and total mass on height are shown in Fig. 10. The only mass of the trusses is shown in all figures (the mass of columns, 1982 kg, isn't evaluated).

In Fig. 8. the four top graphs are distinguished unfavorably by steel input with three of them correspond to the truss with parabolic bottom chord (Fig. 4). That obviously shows the advantage of trusses with parallel chords.

4. The problem of elastic–plastic structure volume optimization

In the case of the monotonically increasing load the mathematical model of the problem of the minimal volume (mass) elastic–plastic structure can be formulated according to the corresponding optimization model of elastic structure, when the plastic strains $\boldsymbol{\varepsilon}_p = [\boldsymbol{\Phi}]^{\mathrm{T}} \boldsymbol{\lambda}$ and additional complementary slackness condition are evaluated

$$\boldsymbol{\lambda}^{\mathrm{T}} \left\{ [G]\,\mathbf{A}_0 - \left[\bar{\Phi}\right]^{\mathrm{T}} \mathbf{S} \right\} = 0 \tag{8}$$

that must correspond to plastic multipliers $\lambda \geq 0$. So, referring to the model (1), it is found such a monotonically increasing load acting on elastic–plastic structure, which corresponds to the requirements of the strength, stiffness and stability, mathematical model of the optimization problem:

$$\begin{aligned} &\text{find} && \min \mathbf{L}^{\mathrm{T}}\mathbf{A}_0 \\ &\text{subject to} && [A]\,\mathbf{S} = \mathbf{F}, \quad \left[\bar{D}\right]\,\mathbf{S} + \left[\bar{\Phi}\right]^{\mathrm{T}} \boldsymbol{\lambda} - [A]^{\mathrm{T}}\,\mathbf{u} = 0, \\ &&& \boldsymbol{\lambda}^{\mathrm{T}} \left\{ [G]\,\mathbf{A}_0 - \left[\bar{\boldsymbol{\Phi}}\right]\,\mathbf{S} \right\} = 0, \quad \boldsymbol{\lambda} \geq 0, \quad [E]\,\mathbf{u} \leq \mathbf{u}^{+}, \\ &&& [G]\,\mathbf{A}_0 - \left[\bar{\Phi}\right]\,\mathbf{S} \geq 0, \quad \mathbf{A}_0 \geq \mathbf{A}_0^{-}, \quad \mathbf{A}_0 \in \Pi. \end{aligned} \tag{9}$$

The search of this nonlinear programming problem solution $\mathbf{S}$, $\mathbf{u}$, $\boldsymbol{\lambda}$, $\mathbf{A}_0$ is very difficult. It is especially hardened by the nonlinear conditions (8). Therefore the problem is solved iteratively, in each iteration selecting cross-sections of bars and solving a simpler problem of nonlinear programming where only additional complementary slackness conditions are nonlinear. For the purpose of admissible (design) set simplification of the problem and its numerical realization, it is needed to eliminate these conditions from the constraints of the problem. This can be done in two ways — by moving them to the objective function (such a possibility is proved in the paper [22] and used in the paper [23]) or eliminating and solving a reduced optimization problem. So in each iteration it is possible to solve such a problem:

$$\begin{aligned} &\text{find} && \min f = \mathbf{L}^{\mathrm{T}}\mathbf{A}_0 + \boldsymbol{\lambda}^{\mathrm{T}} \{[G]\,\mathbf{A}_0 - [\Phi]\,\mathbf{S}\} \\ &\text{subject to} && [A]\,\mathbf{S} = \mathbf{F}, \quad [D]\,\mathbf{S} + [\Phi]^{\mathrm{T}} \boldsymbol{\lambda} - [A]^{\mathrm{T}}\,\mathbf{u} = \mathbf{0}, \\ &&& [G]\,\mathbf{A}_0 - [\Phi]\,\mathbf{S} \geq \mathbf{0}, \quad \boldsymbol{\lambda} \geq \mathbf{0}, \quad [E]\,\mathbf{u} \leq \mathbf{u}^{+}, \\ &&& \mathbf{A}_0 \geq \mathbf{A}_0^{-}, \quad \mathbf{A}_0 \in \Pi \end{aligned} \tag{10}$$

or

$$\begin{aligned} &\text{find} && \min f = \mathbf{L}^{\mathrm{T}}\mathbf{A}_0 \\ &\text{subject to} && [A]\,\mathbf{S} = \mathbf{F}, \quad [G]\,\mathbf{A}_0 - [\Phi]\,\mathbf{S} \geq \mathbf{0}, \\ &&& [D]\,\mathbf{S} + [\Phi]^{\mathrm{T}} \boldsymbol{\lambda} - [A]^{\mathrm{T}}\,\mathbf{u} = \mathbf{0}, \quad \boldsymbol{\lambda} \geq \mathbf{0}, \\ &&& [E]\,\mathbf{u} \leq \mathbf{u}^{+}, \quad \mathbf{A}_0 \geq \mathbf{A}_0^{-}, \ \mathbf{A}_0 \in \Pi. \end{aligned} \tag{11}$$

In the first case is the problem with nonlinear objective function and linear constrains, and in the second case the reduced linear programming problem (RLPP). It's understandable that while solving RLPP, the condition $\lambda_j \left\{ \left[G_j\right] \mathbf{A}_0 - \left[\Phi_j\right] \mathbf{S} \right\} = 0$ of some calculated section won't be satisfied. Therefore in this case for defining the optimal solution it is needed to apply the method of branch and bound, setting additional constraints $\lambda_j \leq 0$ for the recent sections.

Example 3. It is needed to set the cross-sections of the bars of the steel rolled profiles of the optimal framed structure, which calculation scheme is shown in Fig. 3. The height of the truss is $h = 3.3$ m. The columns and the upper chord of the truss are designed from I profiles, and other bars from a rectangular profile tube. The yield strength of the metal $R_y = 275$ MPa, elasticity module $E = 2.1 \times 10^5$ MPa. The requirements of the strength is described via constraints $u_x \leq 5$ cm and $u_y \leq 10$ cm; here u_x—horizontal displacement of column top node, u_y—vertical displacement of truss bottom chord middle node.

Frame bars' optimal cross-sections were determined with the help of the branch and bound method by solving reduced nonlinear programming problems. Such cross-sections of the bars were found: 1 – HEA300; 2 – IPE330; 3—180 × 180 × 6; 4—140 × 140 × 5; 5—90 × 90 × 5; 6—90 × 90 × 4; 7—70 × 70 × 4; 8—80 × 80 × 4; 9—60 × 60 × 5. This solution shows that while designing a structure, in which plastic deformations are allowed, it is possible to reduce only tension 4-th bar cross-section. Minimal mass of the optimal elastic–plastic structure $f = 5178$ kg is only 51 kg smaller than the mass of the optimal elastic structure.

5. Conclusions

1. The problems of steel structure designing are formulated as nonlinear optimization problems. It is demonstrated that elastic and elastic–plastic structures designed from rolled profiles problems are nonlinear discrete optimization problems, whose solutions can be found in an iterative way applying branch and bound method and linear programming.
2. There are proposed three algorithms of optimal bars' structures design, whose relations can be formulated applying the methods of equilibrium and geometrically compatible finite elements.
3. While performed analysis of the bottom chord sketch, as it were various height of the truss, it was determined that the truss with parallel bottom chord (Fig. 3) is more rational, compared with the truss whose bottom chord was formed of quadratic parabolas (Fig. 4).
4. The problem of optimal height determination for truss is formulated and the accomplished calculations determine that height of optimal truss with horizontal bottom chord is $h_{opt} = 4$ m ($h_{opt} = 1/9l$, l—span length) and latter of optimal truss with parabolic bottom chord is 4.5 m ($h_{opt} = 1/8l$.)
5. While fulfilling the analysis of the truss web form and density it was determined the most rational is the triangle web with vertical bars (Fig. 5), while the length of segment is 3.6 m or $1/10 \cdot l$.
6. Elastic-plastic framed structure analysis confirmed the statement that often an optimal structure project is determined not by the strength, but the stiffness, stability and structural requirements.
7. Created mathematical models and solution algorithms for 2D optimization problems can be adopted for solution of 3D optimization problems.

References

[1] Kaneko L, Maier G. Optimum design of plastic structures under displacement's constraints. Comput Methods Appl Mech Eng 1981;27(3):369–92.
[2] Banichuk NV. Introduction to the optimization of the structures. Springer Verlag; 1990.
[3] Majid KI, Optimum design structures. London: Newnes-Butterworths; 1974.
[4] Čyras A. Analysis and optimization of elastoplastic systems. New York: John Wiley & Sons; 1983.
[5] Gutkowski W, editor. Discrete structural optimization. Springer-Verlag; 1997.
[6] Tin-Loi F. Optimum shakedown design under residual displacements constraints. Struct Multidisciplinary Optimiz 2000;19(2):130–9.
[7] Kaliszky S, Logo J. Plastic behaviour and stability constraints in shakedown analysis and optimal design. Struct Multidisciplinary Optimiz 2002;24:118–24.
[8] Karkauskas R. Optimization of geometrically non-linear elastic–plastic structures in the state prior to plastic collapse. J Civil Eng Manage 2007;XIII(3):183–92.
[9] Hayalioglu MS. Optimum design of geometrically non-linear elastic–plastic steel frames via genetic algorithm. Comput Struct 2000;77:527–38.
[10] Hayalioglu MS, Degertekin SO. Design of non-linear steel frames for stress and displacement constraints with semirigid connections via genetic optimization. Struct Multidisciplinary Optimiz 2004;27:259–71.
[11] Zheng QZ, Querin OM, Barton DC. Geometry and sizing optimisation of discrete structure using the genetic programming method. Struct Multidisciplinary Optimiz 2006;31(6):452–61.
[12] Manickarajah D, Xie YM, Steven GP. Optimum design of frames with multiple constraints using an evolutionary method. Comput Struct 2000;74:731–41.
[13] Yuge K, Iwai N, Kikuchi N. Optimization of 2D structures subjected to nonlinear deformations using the homogenization method. Struct optim 1999;17:286–99.
[14] Feng FZ, Kim YH, Yang BS. Application of hybrid optimization techniques for model updating of rotor shafts. Struct Multidisciplinary Optimiz 2006;32(1):67–75.
[15] Merkevičiūte D, Atkočiūnas J. Optimal shakedown design of metal structures under stiffness and stability constraints. J Construct Steel Res 2006;62:1270–5.
[16] Grigusevičius A, Kalanta S. Optimization of elastic–plastic beam structures with hardening using finite element method. Found Civil Environmental Eng 2005;6:31–52.
[17] Janulevičius R, Kalanta S. Optimization of elastic beam structure using linear programming. In: Material of 8th conference of young Lithuanian scientist "Science - Future of Lithuania" held in Vilnius in March 24–25. 2005. p. 194–204 [in Lithuanian].
[18] STR 2.05.08. Design of steel structures. General rules. Design cod of Lithuanian Republic. Vilnius: Lithuanian Department of Environment; 2005 [in Lithuanian].
[19] Kalanta S. The equilibrium finite element in computation of elastic structures. Statyba 1995;1(1):25–47 [in Russian].
[20] Zienkiewicz OC, Taylor RL. Finite element method. Butterworth-Heinemann; 2005.
[21] EN 1993-1-1:2005. Eurocode 3: Design of steel structures — Part 1-1: General rules and rules for buildings; 2005.
[22] Kalanta S. New formulations of optimization problems of elastoplastic bar structures under displacement constraints. Mechanika 1999;5(20):9–16 [in Russian].
[23] Atkočiūnas J, Merkevičiūte D, Venskus A, Skaržauskas V. Nonlinear programming and optimal shakedown design of frames. Mechanika 2007;2(64):27–33.

JOURNAL OF CIVIL ENGINEERING AND MANAGEMENT
2010
16(2): 193–198

INTEGRATED LOAD OPTIMIZATION OF ELASTIC−PLASTIC AXISYMMETRIC PLATES AT SHAKEDOWN

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Received 12 Oct. 2009; accepted 11 Dec. 2009
Extended abstract on enclosed CD-ROM

Abstract. An elastic-plastic axisymmetric steel bending plate subjected to a repeated variable load (RVL) is considered. The solution to the load optimization problem at shakedown is complicated because the stress-strain state of the dissipative systems (e.g. the plate plastic deforming) depends on their loading history. A new algorithm for the load optimization problem combining von Mises and Tresca yield criterion based on the Rosen project gradient method is proposed. The optimization results are obtained by integrating the existing software and that created by the authors.

Keywords: elastic-plastic plates, shakedown, energy principle, Mises and Tresca yield criterion, mathematical programming.

1. Introduction

An elastic – plastic axisymmetric steel bending plate subjected to a repeated variable load (RVL) $\mathbf{F}(t)$ is considered in this paper. The RVL is the system of loads where each of which can independently vary within the time t independent lower and upper bounds of the forces $\mathbf{F}_{inf}$, $\mathbf{F}_{sup}$ ($\mathbf{F}_{inf} \le \mathbf{F}(t) \le \mathbf{F}_{sup}$). An ideal elastic – plastic structure subjected by RVL can exceed its constructive requirements due to a failure caused by its incremental collapse and/or its alternating plasticity. Both cases are usually referred to as cyclic plastic collapse. The shakedown plates are investigated in this paper. The plastic strains $\mathbf{\Theta}_p$ developed in the initial loading cycle produce the residual moments $\mathbf{M}_r$ which ensure the purely elastic response of the plates during the following loading cycles. Load shakedown analysis via numerical and mathematical programming methods is relevant for civil engineering. This has been confirmed by the growing number of investigations in this field (Mróz *et al.* 1995; Weichert *et al.* 2002; Kaliszky and Lógó 2002; Pham 2003; Atkočiūnas *et al.* 2004; Merkevičiūtė and Atkočiūnas 2006; Stonkus *et al.* 2009; Žilinskaitė and Žiliukas 2008).

The solution of load optimization at shakedown is complicated because the stress – strain state of dissipative systems (e. g. the plate deforming) depends on their loading history (Lange-Hansen 1998). The load optimization problem is formulated by integrating extreme energy principles and methods of mathematical programming theory. A new algorithm for the problem combining Mises and Tresca yield criterion for adapted flexural plates optimization based on the Rosen project gradient method is proposed in this paper (Čyras and Atkočiūnas 1984; Atkočiūnas *et al.* 2007a; Atkočiūnas *et al.* 2007b; Atkočiūnas *et al.* 2008). The algorithm is based on the linear Tresca yield criterion. When the optimal solution is obtained, the von Mises yield criterion is applied in the latest step. The proposed algorithm simplifies the numerical solution of the complicated optimization problem when the Mises yield criterion is applied.

2. The main dependencies of a discrete plate

The discrete model of a symmetric round plate in the polar coordinate system $\mathbf{x}=(\rho,\theta)^T$ is obtained by dividing the plate into $k=1,2,...,s$ ($k \in K$) circular finite elements with s_k nodes $l=1,2, s_k=3$ ($l \in L$), where the master nodes are numbered 1 and 3, respectively (see Fig. 1). The polar coordinate system is located in the center of the plate. It is enough to investigate only one radius of the plate because of the internal forces and the displacements do not depend on the coordinate Θ. Consequently, the second order circular element (the internal forces approximated by a second order polynomial) with three nodes, distributed along the radius ρ, is used. The finite elements are numbered along the radius in a consecutive order, starting from the center of the plate.

The circular plate can be subjected by a uniformly distributed load and linearly distributed load located on the plate's boundaries. The properties of the material (modulus of elasticity E and Poisson coefficient ν), thickness t and intensity of the distributed load q remain constant in the whole finite element. The functions

ISSN 1392–3730 print / ISSN 1822–3605 online
doi:10.3846/jcem.2010.20

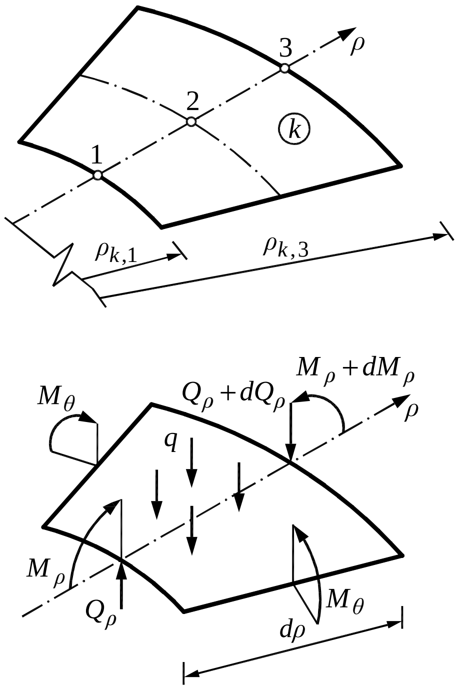

Fig. 1. a) The finite element of a round plate; b) the positive directions of internal forces

of the internal forces distribution can have discontinuities (in the place of master nodes) when the equilibrium of finite elements are applied (Belytschko 1972; Belytschko *et al.* 2000; Gallager 1975; Faccioli and Vitiello 1973; Kalanta 1995) for elastic-plastic plates. Therefore, the finite elements have their own master nodes and sections under investigation and are indexed by the double index kl ($k \in K$, $l \in L$) or by common section index $i = 1, 2, ..., \zeta = s \times s_k$ ($i \in I$) for the discrete plate model. The vectors of internal forces of the finite element k are:

$$\mathbf{M}_k = \left(M_{\rho,k1}, M_{\theta,k1}, M_{\rho,k2}, M_{\theta,k2}, M_{\rho,k3}, M_{\theta,k3}\right)^T$$

$$= \left(\mathbf{M}_{k1}, \mathbf{M}_{k2}, \mathbf{M}_{k3}\right)^T = \left(\mathbf{M}_{kl}\right)^T . \quad (1)$$

Here, $\mathbf{M}_{k1} = (M_{\rho,kl}, M_{\Theta,kl})^T$, and the indexes ρ and Θ denote the radial and angular internal moments, respectively; the positive directions are shown in Fig. 1b.

The bending moments' interpolation function, in applying the finite element k shape function $\mathbf{N}_k(\rho)$ is:

$$\mathbf{M}_k(\rho) = \mathbf{N}_k(\rho)\mathbf{M}_k . \quad (2)$$

The functions (2) do not satisfy the plate element equations:

$$\left(-\frac{d^2}{d\rho^2} - \frac{2}{\rho}\frac{d}{d\rho}\right)M_\rho + \frac{1}{\rho}\frac{d}{d\rho}M_\Theta = q \quad \text{or}$$

$$\mathscr{A}\mathbf{M}(\rho) = q \quad (3)$$

Therefore, equilibrium for the plate elements is assured for the elements and master nodes (Karkauskas 1994). The algebraic equilibrium equation for the finite element is obtained after differentiating the expression (3) which was applied (2):

$$\mathbf{A}_k(\rho)\mathbf{M}_k = q_k , \quad (4)$$

where

$$\mathbf{A}_k(\rho) = \mathscr{A}\mathbf{N}_k(\rho). \quad (5)$$

The separate elements are joined to a system by writing the equilibrium equations for the master nodes of the adjacent elements. Thus, the continuity of the radial moments M_ρ and the shear forces Q_ρ are ensured. The set of plate equilibrium equations while the boundary conditions are applied are:

$$[\mathrm{A}]\mathbf{M} = \mathbf{F} \text{ or } \sum_k [\mathrm{A}_\mathrm{k}]\mathbf{M}_k = \mathbf{F} . \quad (6)$$

The dimension of the matrix $[\mathrm{A}]$ is $(m \times n)$, where $n = \zeta \times 2$. The geometrical equations for the discrete plate model are obtained by applying the virtual stress principle:

$$\delta\mathbf{F}^T\mathbf{u} = \sum_k \int_{A_k} \delta\mathbf{M}_k^T(\rho)\mathscr{D}\mathbf{M}_k(\rho)dA . \quad (7)$$

and by using equations (2) and (6):

$$\sum_k \delta\mathbf{M}_k^T[\mathrm{A}]_k^T\mathbf{u} = \sum_k \delta\mathbf{M}_k^T[\mathrm{D}_\mathrm{k}]\mathbf{M}_k . \quad (8)$$

Here, the symmetric flexibility matrix $[\mathrm{D}_\mathrm{k}]$ of the element k is calculated by the formula:

$$[\mathrm{D}_\mathrm{k}] = \int_{A_k} \mathbf{N}_k^T(\rho)\mathscr{D}\mathbf{N}_k(\rho)dA . \quad (9)$$

The geometrical equations for the finite element are:

$$[\mathrm{A}_\mathrm{k}]^T\mathbf{u} - [\mathrm{D}_k]\mathbf{M}_k = \mathbf{0} \quad (10)$$

and for whole discrete plate model:

$$[\mathrm{A}]^T\mathbf{u} - [\mathrm{D}]\mathbf{M} = \mathbf{0} . \quad (11)$$

Here, $[\mathrm{D}]$ is the quasidiagonal flexibility matrix of the elements. The sequence of the equilibrium equations $[\mathrm{A}]\mathbf{M} = \mathbf{F}$ determine the physical meaning of the components of the displacements vector $\mathbf{u}$.

If the transition to the plastic state is described via the nonlinear Mises-Huber yield condition:

$$M_\rho^2 - M_\rho M_\Theta + M_\Theta^2 \le (M_0)^2 . \quad (12)$$

The plasticity condition is verified in all the nodes of the finite element:

$$\mathbf{M}_{kl}^T[\Pi_{kl}]\mathbf{M}_{kl} \le (M_{0k})^2 , \; k \in K , \; l \in L . \quad (13)$$

Here, $[\Pi_{kl}]$ is the matrix of the Mises-Huber plasticity condition for the bending circular plate

$$\left[\Pi_{kl}\right]=\begin{bmatrix}1 & -0,5\\ -0,5 & 1\end{bmatrix}. \quad (14)$$

The plasticity condition is often expressed in the following form:

$$\varphi_{kl}=\left(M_{0k}\right)^2-\mathbf{M}_{kl}^T\left[\Pi_{kl}\right]\mathbf{M}_{kl}\geq 0\,. \quad (15)$$

The bending moment limit is constant in the entire finite element: $M_{0k}=const$. If the linear Tresca plasticity condition is applied, the equation (15) is described as:

$$\boldsymbol{\varphi}_{kl}=\mathbf{C}_{kl}-\boldsymbol{\Phi}_{kl}\mathbf{M}_{kl}\geq\mathbf{0}\,. \quad (16)$$

The Tresca plasticity condition matrix $\boldsymbol{\Phi}_{kl}$ is:

$$\boldsymbol{\Phi}_{kl}=\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & -1\\ -1 & 1\\ -1 & 0\\ 0 & -1\end{bmatrix} \quad (17)$$

The vector of the limit moments C_{kl} match the matrix $\boldsymbol{\Phi}_{kl}$. For the sake of simplicity, the calculation sections will be indexed as $i=1,2,...,\zeta$, $i\in I$.

3. The main dependencies in the case of cyclic loading

In the practice of engineering, it is necessary to know the deformed state of the plate under plastic deformation just before its cyclic plastic failure (plate geometry, limit moments M_0 and load $\mathbf{F}$ are known) (Kalanta *et al.* 2009; Jankovski and Atkočiūnas 2008). Such a type of structural mechanics problem is referred to as an analysis problem (Cyras 1983). In such a case, it is useful to separate the elastic moments M_e and residual moments M_r: $\mathbf{M}_i=\mathbf{M}_{ei}+\mathbf{M}_{ri}$, $i\in I$. The elastic moments can be calculated by the formula $\mathbf{M}_e=[\alpha]\mathbf{F}$, where the moments influence matrix $[\alpha]$ have the following dimensions $(n\times m)$. When the load $\mathbf{F}(t)$ is a function of time t:

$$\mathbf{M}_i(t)=\mathbf{M}_{ei}(t)+\mathbf{M}_{ri}\,,\ i\in I \quad (18)$$

If RVL is described by their variation boundaries as $\mathbf{F}_{inf}$, $\mathbf{F}_{sup}$, it is possible to determine the possible load combination count p ($j=1,2,...,p$; $j\in J$) and the equation (18) is rewritten as:

$$\mathbf{M}_{ij}=\mathbf{M}_{ei,j}+\mathbf{M}_{ri}\,,\ i\in I \quad (19)$$

The determination of $\mathbf{M}_{ei,j}$ is described in the work (Pham 2003). Then, the Mises-Huber plasticity condition (15) is rewritten as follows:

$$\boldsymbol{\varphi}_{ij}=\left(M_{0k}\right)^2-\mathbf{M}_{ij}^T\left[\Pi_i\right]\mathbf{M}_{ij}\geq 0\,,\ i\in I\,,\ j\in J\,. \quad (20)$$

Thus, in the analysis of shakedown structures, it is the convenient separate residual moments $\mathbf{M}_r$, residual displacements $\mathbf{u}_r$ and deformations $\boldsymbol{\theta}_r=[\mathrm{D}]\mathbf{M}_r+\boldsymbol{\theta}_p$. Then, the equilibrium equations (6) and geometrical equations (11) are described by mentioned terms:

$$[\mathrm{A}]\mathbf{M}_r=\mathbf{0}\ \text{ or }\ \sum_k[\mathrm{A}]_k\mathbf{M}_{rk}=\mathbf{0} \quad (21)$$

and

$$[\mathrm{A}]^T\mathbf{u}_r=[\mathrm{D}]\mathbf{M}_r+\boldsymbol{\theta}_p\,. \quad (22)$$

The components of the plastic deformation's vector $\boldsymbol{\theta}_p=\left(\boldsymbol{\theta}_{p,i}\right)$ are calculated by formula:

$$\boldsymbol{\theta}_{p,i}=\sum_j\left[\nabla\boldsymbol{\varphi}_{ij}\left(\mathbf{M}_{ei,j}+\mathbf{M}_{ri}\right)\right]^T\lambda_{ij}\,,$$

$$\lambda_{ij}\geq 0\,,\ i\in I\,,\ j\in J\,. \quad (23)$$

Here, λ_{ij} is the plastic multiplier vector; $\left[\nabla\varphi_{ij}\right]$ – a matrix composed from the gradients of the plasticity conditions (20).

4. The mathematical models of the analysis problem

The static formulation of the analysis problem is based on the additional energy minimum principle and in the case of Mises plasticity conditions:

find

$$\min\frac{1}{2}\sum_k\mathbf{M}_{rk}^T\left[\mathrm{D}_k\right]\mathbf{M}_{rk}\,, \quad (24)$$

when

$$\sum_k\left[\mathrm{A}_k\right]\mathbf{M}_{rk}=\mathbf{0}\,,k\in K\,, \quad (25)$$

$$\varphi_{ij}=\left(M_{0i}\right)^2-\left(\mathbf{M}_{ei,j}+\mathbf{M}_{ri}\right)^T\left[\Pi_i\right]\left(\mathbf{M}_{ei,j}+\mathbf{M}_{ri}\right)\geq 0\,,$$

$$i\in K\,,\ j\in J\,. \quad (26)$$

The optimal solution of the problem (24)–(26) is $\mathbf{M}_r^*$.

The kinematic formulation of the problem under analysis is created in accordance with the mathematical programming duality theory:

find

$$\max\left\{\begin{array}{l}-\dfrac{1}{2}\mathbf{M}_{rk}^T\left[\mathrm{D}_k\right]\mathbf{M}_{rk}-\sum_i\sum_j\lambda_{ij}\left[\nabla\varphi_{ij}\right]\mathbf{M}_{ri}\\ -\sum_i\sum_j\lambda_{ij}\left[\left(M_{0i}\right)^2-\mathbf{M}_{ij}^T\left[\Pi_i\right]\mathbf{M}_{ij}\right]\end{array}\right\}, \quad (27)$$

when

$$\left[\mathrm{D}_k\right]\mathbf{M}_{rk}+\sum_j\left[\nabla\boldsymbol{\varphi}_{kj}\right]^T\boldsymbol{\lambda}_{kj}-\left[\mathrm{A}_k\right]^T\mathbf{u}_r=\mathbf{0}\,, \quad (28)$$

$$\boldsymbol{\lambda}_{kj}\geq\mathbf{0}\,,\ k\in K\,,\ i\in I\,,\ j\in J\,. \quad (29)$$

The optimal solution of the kinematic formulation (27)–(29) is $\mathbf{M}_r^*$, $\boldsymbol{\lambda}_{kj}^*$, $\mathbf{u}_r^*$.

In the case of the Tresca plasticity condition, only equation (26) should be changed:

$$\boldsymbol{\varphi}_{ij} = \mathbf{C}_i - [\Phi_i](\mathbf{M}_{ei,j} + \mathbf{M}_{ri}) \geq \mathbf{0} . \quad (30)$$

The vector $\mathbf{C}_i$ contains the limit moments of the corresponding finite element.

5. The influence matrixes of the residual displacements and residual moments

If the solution of the static (24)–(26) and kinematic (27)–(29) analysis problem is unknown, then it can be obtained from the nonlinear set of equations:

$$[\mathrm{A}]\mathbf{M}_r = \mathbf{0} , \quad (31)$$

$$\boldsymbol{\varphi}_{ij} = (M_{0k})^2 - \mathbf{M}_{ij}^T[\Pi_i]\mathbf{M}_{ij} , \quad (32)$$

$$\lambda_{ij}\left[(M_{0k})^2 - \mathbf{M}_{ij}^T[\Pi_i]\mathbf{M}_{ij}\right] = 0 , \ \lambda_{ij} \geq 0 , \quad (33)$$

$$[\mathrm{D}]\mathbf{M}_r + \sum_j [\nabla\boldsymbol{\varphi}_j]^T \boldsymbol{\lambda}_j - [\mathrm{A}]^T\mathbf{u}_r = \mathbf{0} , \quad (34)$$

$$\boldsymbol{\lambda}_j \geq (\lambda_{ij}), \ i \in I , \ j \in J . \quad (35)$$

The equation set is composed of the constraints of the static formulation problem (24)–(26) and the Kuhn–Tucker conditions (Bazaraa *et al.* 2004). When the plastic deformations $\boldsymbol{\theta}_p^*$ are known, then from the set of equations

$$\boldsymbol{A}\boldsymbol{M}_r^* = \mathbf{0} ,$$

$$\boldsymbol{D}\boldsymbol{M}_r^* + \boldsymbol{\theta}_p^* - \boldsymbol{A}^T\boldsymbol{u}_r^* = \mathbf{0}$$

it is possible to find the right values of $\mathbf{M}_r^*$ and $\mathbf{u}_r^*$:

$$\mathbf{u}_r^* = \left([\mathrm{A}][\mathrm{D}]^{-1}[\mathrm{A}]^T\right)^{-1}[\mathrm{A}][\mathrm{D}]^{-1}\boldsymbol{\theta}_p^* = [\overline{\mathbf{H}}]\boldsymbol{\theta}_p^* , \quad (36)$$

$$\mathbf{M}_r^* = \left[[\mathrm{D}]^{-1}[\mathrm{A}]^T\left([\mathrm{A}][\mathrm{D}]^{-1}[\mathrm{A}]^T\right)^{-1}[\mathrm{A}][\mathrm{D}]^{-1}\right]\boldsymbol{\theta}_p^* ;$$

$$\mathbf{M}_r^* = [\overline{\mathbf{G}}]\boldsymbol{\theta}_p^* . \quad (37)$$

The vectors $\mathbf{u}_r^*$ and $\mathbf{M}_r^*$, calculated by formulas (36) and (37), respectively, coincide with the optimal ones calculated by the mathematical models (24)–(26) and (27)–(29).

The residual displacement and residual moments influence matrixes $[\overline{\mathrm{H}}]$ and $[\overline{\mathrm{G}}]$, and in the case of Tresca plasticity conditions, do not depend on internal forces $\mathbf{M}_j$:

$$\mathbf{u}_r^* = [\overline{\mathrm{H}}][\Phi]^T\boldsymbol{\lambda}^* = [\mathrm{H}]\boldsymbol{\lambda}^* , \ \mathbf{M}_r^* = [\overline{\mathrm{G}}][\Phi]^T\boldsymbol{\lambda}^* = [\mathrm{G}]\boldsymbol{\lambda}^* . \quad (38)$$

This feature has an important significance for the creation of the mathematical models for the load optimization problem: initially, the Tresca yield condition is applied and only in the latest step is the Mises plasticity criterion applied.

6. The algorithm of RVL optimization

The shakedown plate is safe in respect to plastic collapse, but it can exceed the requirements of serviceability (i.e. stiffness constraints). Therefore, in the mathematical model of the plate load, optimization should not only be included in the requirements of the strength (plasticity), but the constraints for displacements, too. The mathematical model in the case of Tresca plasticity conditions is:

find

$$\max \left(\mathbf{T}_{sup}^T\mathbf{F}_{sup} + \mathbf{T}_{inf}^T\mathbf{F}_{inf}\right) \quad (39)$$

when

$$\boldsymbol{\varphi}_{ij} = \mathbf{C}_i - [\Phi_i](\mathbf{M}_{ei,j} + [\mathrm{G}]\boldsymbol{\lambda}) \geq \mathbf{0} , \quad (40)$$

$$\boldsymbol{\lambda}_{ij}\left[\mathbf{C}_i - [\Phi_i](\mathbf{M}_{ei,j} + [\mathrm{G}]\boldsymbol{\lambda})\right] = 0 , \quad (41)$$

$$\boldsymbol{\lambda} = (\boldsymbol{\lambda}_{ij}), \ i \in I , \ j \in J \quad (42)$$

$$\mathbf{u}_{min} \leq [\mathrm{H}]\boldsymbol{\lambda} + \mathbf{u}_{e,inf} , \quad (43)$$

$$[\mathrm{H}]\boldsymbol{\lambda} + \mathbf{u}_{e,sup} \leq \mathbf{u}_{max} . \quad (44)$$

Here, $\mathbf{u}_{e,sup}$ and $\mathbf{u}_{e,inf}$ are the maximal and minimal elastic displacements, respectively. They, summarized together with the residual displacements $\mathbf{u}_r$, should not exceed the prescribed maximal and minimal displacements boundaries, $\mathbf{u}_{max}$ and $\mathbf{u}_{min}$. The solution of the optimization problem is $\mathbf{F}_{sup}^*$, $\mathbf{F}_{inf}^*$, $\boldsymbol{\lambda}^*$. The algorithm of the load optimization problem illustrating the switch from Tresca to the Mises plasticity condition is shown in Fig. 2.

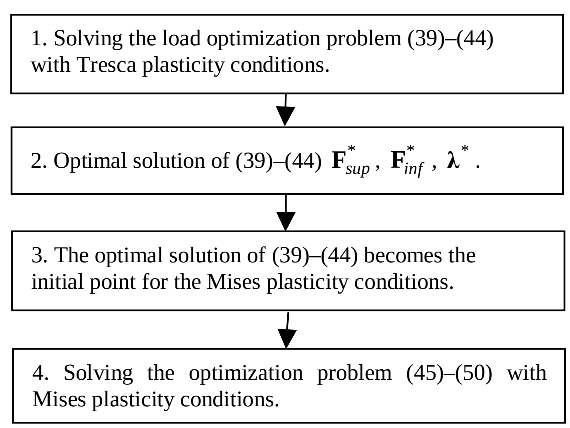

Fig. 2. The algorithm of load optimization with Tresca and Mises plasticity conditions

The mathematical model of the load optimization problem in the case of Mises plasticity conditions is composed using the influence matrixes $[\mathrm{G}]$ and $[\mathrm{H}]$:

find

$$\max \left(\mathbf{T}_{sup}^T \mathbf{F}_{sup} + \mathbf{T}_{inf}^T \mathbf{F}_{inf}\right) \quad (45)$$

when

$$\varphi_{ij} = (M_{0i})^2 - \left(\mathbf{M}_{ei,j} + [\mathrm{G}]\boldsymbol{\lambda}\right)^T [\Pi_i] \left(\mathbf{M}_{ei,j} + [\mathrm{G}]\boldsymbol{\lambda}\right) \geq 0, \quad (46)$$

$$\lambda_{ij}\left[(M_{0i})^2 - \left(\mathbf{M}_{ei,j} + [\mathrm{G}]\boldsymbol{\lambda}\right)^T [\Pi_i] \left(\mathbf{M}_{ei,j} + [\mathrm{G}]\boldsymbol{\lambda}\right)\right] = 0, \quad (47)$$

$$\lambda_{ij} > 0, \ \boldsymbol{\lambda} = (\lambda_{ij}), \ i \in I, \ j \in J \quad (48)$$

$$\mathbf{u}_{min} \leq [\mathrm{H}]\boldsymbol{\lambda} + \mathbf{u}_{e,inf}, \quad (49)$$

$$[\mathrm{H}]\boldsymbol{\lambda} + \mathbf{u}_{e,sup} \leq \mathbf{u}_{max}. \quad (50)$$

The graphical illustration of the switch from Tresca to Mises plasticity conditions is shown in Fig. 3.

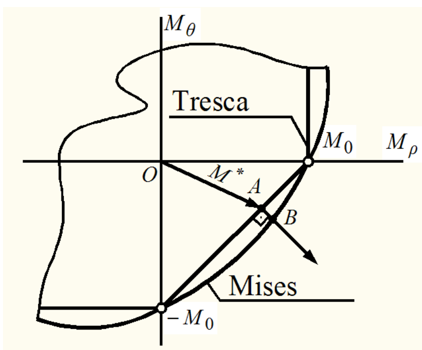

Fig. 3. The fragment of the switch from Tresca plasticity conditions to Mises plasticity conditions

7. Numerical example

The proposed calculation technique is illustrated by the example of a circular plate with a hole in the middle (Fig. 4). The supports are applied in the outside boundary of plate.

Radius of plate $R = 1.0$ m, height $h = 0.025$ m, diameter of hole $d = 0.30$ m. The material – steel, $E = 210$ GPa, $\nu = 0.3$, $\sigma_y = 235$ MPa. The limit moment of the plate $M_0 = \frac{1}{4}\sigma_y t^2 = 36.719$ kNm.

The outside boundary of the plate is loaded by the uniformly distributed linear moment $M = 5.0$ kNm/m, and the surface of the plate is subjected to a uniformly distributed load *q*, which is an unknown of the optimization problem. The displacement variations have boundaries which are $u_{min} = 0.$ m, $u_{max} = 0.037$ m in the place of the hole. When the problem (39)–(44) was solved, the optimal load of $q^* = 131.246$ kPa was obtained. In the case of the Mises plasticity condition, the following more optimal solution was obtained: $q^* = 140.747$ kPa.

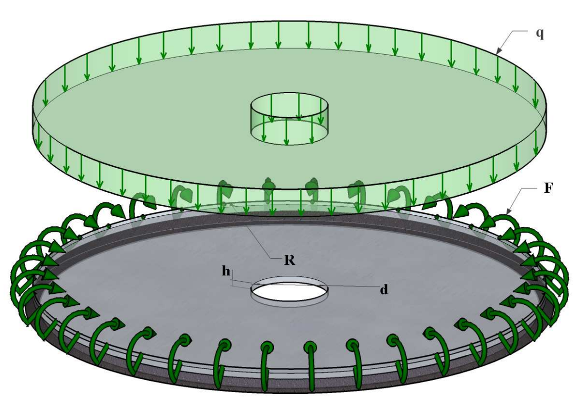

Fig. 4. The geometry of the round plate and boundary conditions

8. Conclusions

1. The influence matrixes of residual moments and displacements do not depend on the residual moments of $\mathbf{M}_r$.

2. In the case of Mises plasticity conditions, the influence matrixes should be formulated using the gradients of plasticity conditions, which themselves depend on $\mathbf{M}_r$. The main load optimization problem, in the case of Mises, becomes practically not realizable, even with applied computer algebra methods.

3. One of the possible resolutions of the load optimization problem with a Mises plasticity condition is the application of an analogous problem solution obtained with Tresca plasticity conditions.

References

Atkočiūnas, J.; Jarmolajeva, E.; Merkevičiūtė, D. 2004. Optimal shakedown loading for circular plates, *Structural and Multidisciplinary Optimization* 27(3): 178–188.

Atkočiūnas, J.; Merkevičiūtė, D.; Venskus, A.; Skaržauskas, V. 2007a. Nonlinear programming and optimal shakedown design of frames, *Mechanika* 64(2): 27–33.

Atkočiūnas, J.; Rimkus, L.; Skaržauskas, V.; Jarmolajeva, E. 2007b. Optimal shakedown design of plates, *Mechanika* 67(5): 14–23.

Atkočiūnas, J.; Merkevičiūtė, D.; Venskus, A. 2008. Optimal shakedown design of bar systems: Strength, stiffness and stability constrains, *Computers & Structures* 86(17–18): 1757–1768.

Bazaraa, M. S.; Sherali, H. D.; Shetty, C. M. 2004. *Nonlinear programming: theory and algorithms*. New York, Brijbasi Art Press Ltd., John Wiley & Sons, Inc. 652 p.

Belytschko, T. 1972. Plane stress shakedown analysis by finite elements, *International Journal of Mechanical Sciences* 14: 619–625.

Belytschko, T.; Liu, W. K.; Moran, B. 2000. *Nonlinear finite elements for continua and structures*. New York: John Wiley & Sons Ltd.

Cyras, A. A. 1983. *Mathematical models for the analysis and optimization of elastoplastic structures*. Chichester: Ellis Horwood Lim. 121 p.

Čyras, A.; Atkočiūnas, J. 1984. Mathematical model for the analysis of elastic-plastic structures under repeated-variable loading, *Mechanics Research Communications* 11: 353–360.

Faccioli, E.; Vitiello, E. 1973. A finite element linear programming method for the limit analysis of thin plates, *International Journal for Numerical Methods in Engineering* 5: 311–325.

Gallager, R. H. 1975. *Finite element analysis. Fundamentals*. Englewood Clifts: Prentice-Hall Inc.

Jankovski, V.; Atkočiūnas, J. 2008. *MATLAB* implementation in direct probability design of optimal steel trusses, *Mechanika* 74(6): 30–37.

Kalanta, S. 1995. Равновесные конечные элементы в расчётах упругих конструкций [The equilibrium finite elements in computation of elastic structures], *Statyba* [Civil Engineering] 1: 25–47.

Kalanta, S.; Atkočiūnas, J.; Venskus, A. 2009. Discrete optimization problems of the steel structures, *Engineering Structures* 31(6): 1298–1304.

Kaliszky, S.; Lógó, J. 2002. Plastic behaviour and stability constraints in the shakedown analysis and optimal design of trusses, *Structural and Multidisciplinary Optimization* 24(2): 118–124.

Karkauskas, R.; Krutinis, A.; Atkočiūnas, J.; Kalanta, S.; Nagevičius, J. 1994. *Statybinės mechanikos uždavinių sprendimas kompiuteriu* [Solution of Structural Mechanics Problems by Computers]. Vilnius: Mokslo ir enciklopedijų l-kla [Vilnius: Science and Encyclopaedic Publishing House]. 264 p.

Lange-Hansen, P. 1998. *Comparative study of upper bound methods for the calculation of residual deformations after shakedown*. Lygby, Denmark.

Merkevičiūtė, D.; Atkočiūnas, J. 2006. Optimal shakedown design of metal structures under stiffness and stability constraints, *Journal of Constructional Steel Research* 62(12): 1270–1275.

Mróz, Z.; Weichert, D.; Dorosz, S.; Editors. 1995. *Inelastic Behavior of Structures under Variable Loads*. Dordrecht: Kluwer Academic Publishers.

Pham, D. C. 2003. Plastic collapse of a circular plate under cyclic loads, *International Journal of Plasticity* 19: 547–559.

Stonkus, R.; Leonavičius, M.; Krenevičius, A. 2009. Cracking threshold of the welded joints subjected to high-cyclic loading, *Mechanika* 76(2): 5–10.

Weichert, D.; Maier, G.; Editors. 2002. *Inelastic Behavior of Structures under Variable Repeated Loads*. New York, Vienna: Springer.

Žilinskaitė, A.; Žiliukas, A. 2008. General deformation flow theory, *Mechanika* 70(2): 11–15.

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008

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Inhalt

Optimal shakedown design of steel structures

J. Atkočiūnas, D. Merkevičiūtė, A. Venskus, L. Rimkus

1 Introduction

Steel structures, which undergo plastic strains and are subjected by variable repeated load, are considered in the paper. Under repeated loading a structure can lose its serviceability because of its progressive plastic failure or because of alternating strain (usually both cases are called cyclic–plastic collapse). The third case is also possible, when the structure adapts to existing load and further behaves only elastically. For civil engineering, calculation of any complexity elastic–plastic structures subjected by variable repeated load is relevant. Growing number of scientific works dedicated to adapted structure calculation shows importance of these researches [1]-[9]. But there is especially small number of works concerning optimization of adapted structures under stiffness and stability constraints. That had influence on the topic of this paper: optimal shakedown design of structures, subjected by variable repeated load, under stiffness and stability constraints. Solution of structure optimization problems at shakedown is complicated as stress–strain state of dissipative systems depends on loading history [10]-[14]. These difficult optimization problems are implemented applying extremum energy principles, theory of mathematical programming [15]. New iterative algorithm based on Rosen project gradient method is created [16]. Numerical examples of frame, truss and plate design are presented (Fig 3).

2 General mathematical models of optimization problems

The mathematical models presented in Table 1 are applied for optimization of bending plates, frames and trusses at shakedown in this research. Stiffness conditions (4), (8) are realized by the restriction of structure nodal displacements $\boldsymbol{u}$. Non-linear mathematical programming is applied for problem solution. The Rozen project gradient method is applied to solve the cyclically loaded non-linear shakedown steel structures strain evaluation.

Table 1. Mathematical models of structure optimal design problem

Linear yield conditions	Non–linear yield conditions
find $\min \psi(\boldsymbol{S}_0) = \min \boldsymbol{L}^{\mathrm{T}}\boldsymbol{S}_0$, (1) subject to $\boldsymbol{\varphi}_j = \boldsymbol{S}_0 - \boldsymbol{\Phi}(\boldsymbol{G}\boldsymbol{\lambda} + \boldsymbol{S}_{ej}) \geq \boldsymbol{0}$, (2) $\boldsymbol{\lambda}_j^{\mathrm{T}}\boldsymbol{\varphi}_j = 0$, $\boldsymbol{\lambda}_j \geq \boldsymbol{0}$, (3) $\boldsymbol{\lambda} = \sum_j \boldsymbol{\lambda}_j$, $j \in J$, $\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf}$, $\boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max}$. (4)	find $\min \boldsymbol{L}^{\mathrm{T}}\boldsymbol{S}_0$, (5) subject to $\min \tilde{F}(S_r) = \min \frac{1}{2}\boldsymbol{S}_r^{\mathrm{T}}\boldsymbol{D}\,\boldsymbol{S}_r$, (6) $\boldsymbol{A}\boldsymbol{S}_r = \boldsymbol{0}$, $\boldsymbol{\varphi}_j = \boldsymbol{C} - \boldsymbol{f}_j(\boldsymbol{S}_r + \boldsymbol{S}_{ej}) \geq \boldsymbol{0}$, $j \in J$, $\boldsymbol{C} = \boldsymbol{C}(\boldsymbol{S}_0)$, $\boldsymbol{S}_0 \geq \boldsymbol{0}$, (7) $\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf}$, $\boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max}$. (8)

3 *Frame volume minimization at shakedown.*

Mathematical model of the problem ($\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$, σ_{yk}, L_k, $k \in K$ are known) reads:

find

$$\min \sum_k L_k A_k \tag{9}$$

subject to

$$\boldsymbol{\varphi}_j = \boldsymbol{M}_0 - \boldsymbol{\Phi}\left(\boldsymbol{G}\boldsymbol{\lambda} + \boldsymbol{S}_{ej}\right) \geq \boldsymbol{0}; \tag{10}$$

$$\sum_{j=1}^{p} \boldsymbol{\lambda}_j^{\mathrm{T}}\left[\boldsymbol{M}_0 - \boldsymbol{\Phi}\left(\boldsymbol{G}\boldsymbol{\lambda} + \boldsymbol{S}_{ej}\right)\right] = 0, \quad \boldsymbol{\lambda}_j \geq \boldsymbol{0}, \quad \boldsymbol{\lambda} = \sum_{j=1}^{p} \boldsymbol{\lambda}_j, \quad j \in J; \tag{11}$$

$$A_k \geq A_{k,min}, \quad k \in K \tag{12}$$

$$\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf}, \qquad \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max} \tag{13}$$

In the problem (9)–(13) unknowns are the cross–sectional areas A_k, $k \in K$ of frame elements and vectors of plasticity multipliers $\boldsymbol{\lambda}_j \geq \boldsymbol{0}$, $j \in J$ ($M_{0k} = \sigma_{yk} W_{pl,k} = \xi(\sigma_{yk}, A_k)$, $N_{0k} = \sigma_{yk} A_k$ are functions of the cross–sectional area A_k and material yield limit σ_{yk}. Depending on the cross-sectional shape various yield conditions can be considered. In this paper, focus is placed on yield conditions for rolled I, H and hollow square steel sections (Fig.1):

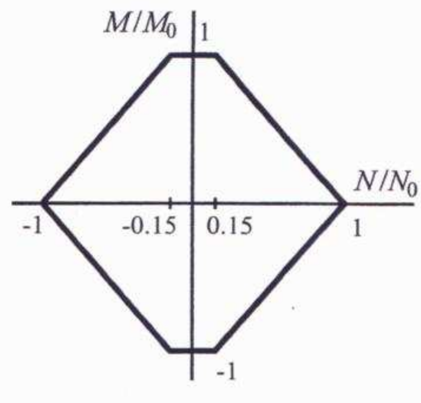

Fig 1. Linear yield conditions

Lower bounds of the cross–sectional areas $A_{k,\,min}$ are included into constructive constraints (12). Limit moments $\boldsymbol{M}_0$, influence matrices $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{G}$, $\boldsymbol{H}$ are related with design variables A_k, $k \in K$; listed matrices are recalculated during the solution process of the problem (9)–(13).

4 *Evaluation of bar stability*

For trusses stability conditions (besides strength and stiffness requirements) are related with recommendations of EC3, when admissible forces of compressive bars are obtained by reduction of their material yield limit σ_y (vector of limit forces $\boldsymbol{N}_0$ ($N_{0j} = \sigma_{yk} A_k$, $k \in K$) is substituted by $\boldsymbol{N}_{0,cr}$). Then yield conditions of discretized truss read:

$$\boldsymbol{\varphi}_{max} = \boldsymbol{N}_0 - \boldsymbol{N}_r - \boldsymbol{N}_{e,max} \geq \mathbf{0}, \ \boldsymbol{\varphi}_{min} = \boldsymbol{N}_{0,cr} + \boldsymbol{N}_r + \boldsymbol{N}_{e,min} \geq \mathbf{0}. \tag{14}$$

Here $\boldsymbol{N}_{e,\max} = \boldsymbol{\alpha}_{sup} \boldsymbol{F}_{sup} + \boldsymbol{\alpha}_{inf} \boldsymbol{F}_{inf}$, $\boldsymbol{N}_{e,\min} = \boldsymbol{\alpha}_{sup} \boldsymbol{F}_{inf} + \boldsymbol{\alpha}_{inf} \boldsymbol{F}_{sup}$ are vectors of minimal and maximal values of elastic axial forces; $\boldsymbol{N}_0 = (N_{0k})^{\mathrm{T}}$, $\boldsymbol{N}_{0,cr} = (N_{0k,cr})^T$, $N_{0,k} = \sigma_{yk} A_k$, $N_{0,k,cr} = \varphi_k \sigma_{yk} A_k$, $k \in K$. $N_{0,cr,k}$ are calculated according to the formulas:

$$N_{0,cr,k} = \varphi_k N_{0,k}, \ \varphi_k = \frac{1}{\Phi_k + \left[\Phi_k^2 - \bar{\lambda}_k^2\right]^{0.5}}, \tag{15}$$

when

$$\Phi_k = 0.5\left(1 + a\left(\bar{\lambda}_k - 0.2\right) - \bar{\lambda}_k^2\right), \ \bar{\lambda}_k = \frac{\lambda_k}{\lambda_{1k}} \sqrt{\beta_A} = \frac{\lambda_k}{\pi \, [E_k / \sigma_{y,k}]^{0.5}} \sqrt{\beta_A}, \ k \in K. \tag{16}$$

Here E_k is an elasticity modulus of the *k*–th bar; $\lambda_k = L_k / i_k$ is bar slenderness, where i_k is the radius of gyration of the *k*–th bar. In the case of bar under pure compression $\beta_A = 1$; value of imperfection factor a, depends on the shape of cross–sections and properties of applied material. Possible failure because of stability lost is not evaluated when $\boldsymbol{N}_{0,cr} = \boldsymbol{N}_0$.

5 *The problem of truss volume minimization*

Project of minimum volume of adapted truss is determined (when load variation bounds $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$, material yield limit σ_{yk} and lengths L_k of all k ($k \in K$) elements are prescribed) by solving the following problem: *find truss of minimum volume* $V = \sum_k L_k A_k$ *($k \in K$), satisfying requirements of strength, stiffness and stability.*
Mathematical model of non–linear problem reads:

find

$$\min \sum_k L_k A_k \tag{17}$$

subject to

$$\boldsymbol{\varphi}_{max}(A) = \boldsymbol{N}_0 - \boldsymbol{G}\boldsymbol{\Theta}_p - \boldsymbol{N}_{e,max} \geq \mathbf{0}, \ \boldsymbol{\varphi}_{min}(A) = \boldsymbol{N}_{0,cr} + \boldsymbol{G}\boldsymbol{\Theta}_p + \boldsymbol{N}_{e,min} \geq \mathbf{0}, \tag{18}$$

$$\boldsymbol{N}_0 = (N_{0,k})^{\mathrm{T}}, \ \boldsymbol{N}_{0,cr} = (N_{0,k,cr})^{\mathrm{T}}, \ N_{0,k} = \sigma_{yk} A_k, \ N_{0,k,cr} = \varphi_k \sigma_{yk} A_k, \tag{19}$$

$$A_k \geq A_{k,min}, \quad k \in K, \tag{20}$$

$$\boldsymbol{\Theta}_p = \boldsymbol{\lambda}_{max} - \boldsymbol{\lambda}_{cr}, \tag{21}$$

$$\boldsymbol{\lambda}_{max}^{\mathrm{T}} \boldsymbol{\varphi}_{max} = 0, \ \boldsymbol{\lambda}_{cr}^{\mathrm{T}} \boldsymbol{\varphi}_{min} = 0, \ \boldsymbol{\lambda}_{max} \geq \mathbf{0}, \ \boldsymbol{\lambda}_{cr} \geq \mathbf{0}, \tag{22}$$

$$\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf}, \quad \boldsymbol{u}_{r,sup} \leq \boldsymbol{u}_{r,max}. \tag{23}$$

Unknowns are cross–sectional areas A_k, $k \in K$ of bars and vectors of plastic multipliers $\boldsymbol{\lambda}_{max}$, $\boldsymbol{\lambda}_{cr}$. Stiffness constraints (23) are realized via restriction of nodal displacements. Influence matrices $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{H}$ and $\boldsymbol{G}$ depend on design variable A_k, $k \in K$. Possibility to evaluate load combinations, change of temperature and distortions makes mathematical model (17)–(23) important for practical design.

6 *Load optimization problem of bending plates*

General mathematical model of load optimization problem for bending plates at shake-down reads:

find

$$\min \left\{ \boldsymbol{T}_{sup}^{T} \boldsymbol{F}_{sup} + \boldsymbol{T}_{inf}^{T} \boldsymbol{F}_{inf} \right\} = W \tag{24}$$

subject to

$$\text{minimize } \frac{1}{2} \boldsymbol{S}_r^{\cdot T} \left[\widetilde{D}\right] \boldsymbol{S}_r^{\cdot}, \tag{25}$$

$$\text{when } \varphi_{kl,j} = C_k - \boldsymbol{S}_{kl,j}^{T} \left[\Phi\right] \boldsymbol{S}_{kl,j} \geq 0,\ C_k = \left(S_{0k}\right)^2,\ \boldsymbol{S}_{kl,j} = \boldsymbol{S}_{ekl,j} + \boldsymbol{S}_{rkl},\ \ \boldsymbol{S}_r = \left[B\right]^T \boldsymbol{S}_r^{\cdot}, \tag{26}$$

$$\boldsymbol{F}_{inf} \geq \mathbf{0},\quad \boldsymbol{F}_{sup} \geq \mathbf{0}, \tag{27}$$

and

$$\boldsymbol{u}_{r,min} \leq \boldsymbol{u}_{r,inf} = \min \left[H_\theta\right] \boldsymbol{\Theta}_p,\ \boldsymbol{u}_{r,sup} = \max \left[H_\theta\right] \boldsymbol{\Theta}_p \leq \boldsymbol{u}_{r,max},\ k \in K,\ l \in L,\ j \in J. \tag{28}$$

In the problem unknowns are $\boldsymbol{F}_{inf}$, $\boldsymbol{F}_{sup}$, $\boldsymbol{\lambda}_j$. Therefore the residual deflections variation bounds $\boldsymbol{u}_{r,inf}$, $\boldsymbol{u}_{r,sup}$ evaluation problem (29)-(31) needs to be solved:

$$\begin{matrix}\text{maximize}\\ \text{minimize}\end{matrix} \left[\widetilde{H}_i\right] \widetilde{\boldsymbol{\lambda}} = \begin{bmatrix} u_{ri,sup} \\ u_{ri,inf} \end{bmatrix} \tag{29}$$

subject to

$$-\left[\widetilde{B}_\lambda\right] \widetilde{\boldsymbol{\lambda}} = \left[\widetilde{B}_r\right] \boldsymbol{S}_r^{\cdot},\quad \widetilde{\boldsymbol{\lambda}} \geq 0, \tag{30}$$

$$\widetilde{\boldsymbol{\lambda}} = \left(\widetilde{\boldsymbol{\lambda}}_j\right),\ \sum_j \widetilde{\boldsymbol{\lambda}}_j^T \widetilde{\boldsymbol{C}} \leq \widetilde{D}_{max}. \tag{31}$$

During structure adaptation process the energy is dissipated, which the upper bound D_{max} can be calculated by Koiter's suggested formula [17]. The fictitious structure method allows to determine more exact the energy dissipation bound magnitude $\widetilde{D}_{max}$ and obtain improved residual displacement variation bounds $\boldsymbol{u}_{r,inf}$, $\boldsymbol{u}_{r,sup}$. Here the notation of plasticity multipliers $\widetilde{\boldsymbol{\lambda}}$ is compatible with notation $\widetilde{D}_{max}$.

7 *Numerical examples*

Example 1. Numerical illustration of two-storey frame volume minimization at shakedown is presented in Fig 3a. Frame is discretized by using equilibrium finite elements. For columns and for beams are used finite elements with six degrees of freedom under bending and axial loading. In case of beams subjected by distributed load, elements with seven degrees of freedom with linear displacements of central node can be used (Fig 2).

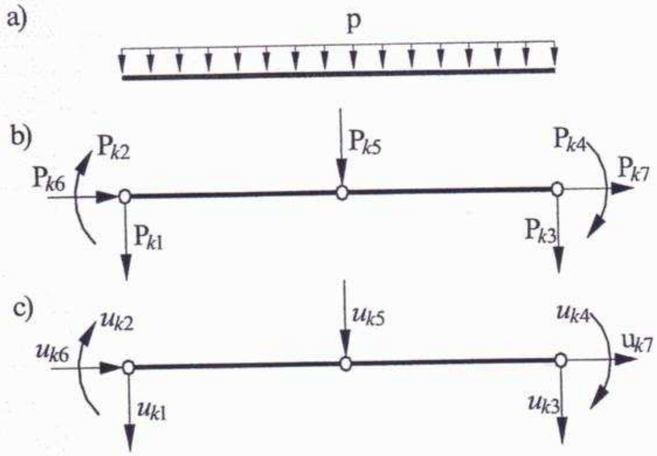

Fig 2. Finite element subjected by distributed load with linear displacements of central node: a) external load; b) generalised forces; c) nodal displacements

The later elements [18] exactly models the stress and strain field of beams and allow to compute directly the middle section displacements of beams. It creates conditions for decrease the number of unknown of optimization problem and obtaining information, whish is necessary to be analyzed later.

Example 2 and 3. Calculations results of volume minimization problem of 9–bar and 20–bar (in case of moving load) trusses are showed in Fig 4a and Fig 4b.

Example 4. Incremental analysis example of bending annular plate is presented in Fig 3b.

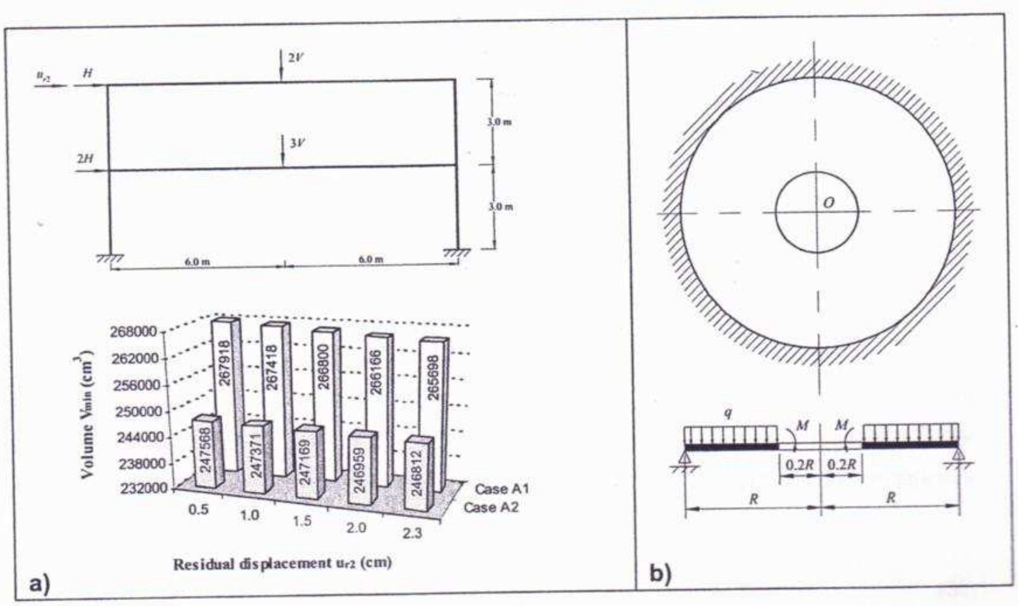

Fig 3. Objects of numerical examples

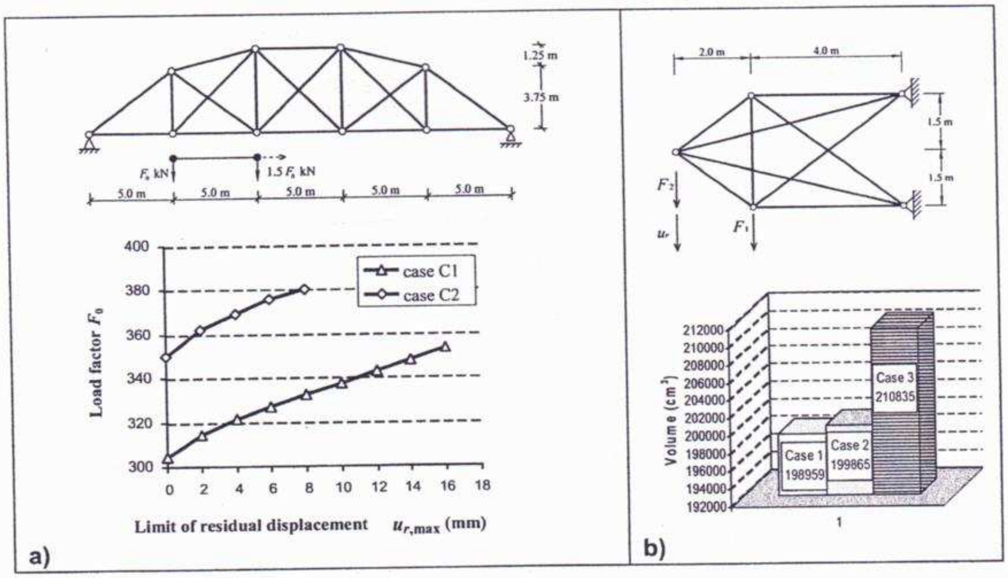

Fig 4. Objects of numerical examples

Example 5. The hinge-fixed perfectly elastic-plastic circular radii R plate is under consideration (Fig. 5). The plate limit bending moment $M_0 = const$ is prescribed (sandwich cross-section), the Poisson ratio is equal to 0.3. The plate is subjected to the cyclic uniformly distributed load q $(0 \le q \le q_{sup})$ and that of uniformly applied bending moment, distributed onto the outer contour M $(0 \le M \le M_{sup})$ Fig. 5. The load variation upper bounds q_{sup} and M_{sup} are to be determined taking into account the plate middle point deflection restriction: $0 \le u_{r1,max} = 0.84\, M_0 R^2 / K$. The load optimization problem of plate at shakedown is realized via the mathematical model (24)-(28).

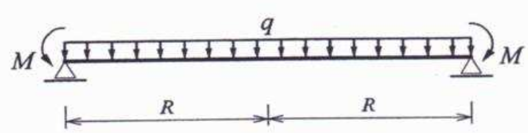

Fig 5. Hinge-fixed circular plate

The equilibrium finite elements are applied for discretization, Huber-Mises yield conditions (Fig 6) are verified for all $(p = 3)$ elastic stresses locus apices. The solution process is iterative. When $q^{v=1} = 5.3881\, M_0 R^{-2}$, $M^{v=1} = 0.8882\, M_0$ the plate analysis problem (25)-(27) is realized applying the Rozen project gradient method. The optimality criterion's mathematical-mechanical interpretation (32)

$$\boldsymbol{\lambda}^* = \left(\left[\nabla\varphi(\boldsymbol{x}^*)\right]\left[\nabla\varphi(\boldsymbol{x}^*)\right]^T\right)^{-1}\left[\nabla\varphi(\boldsymbol{x}^*)\right]\nabla F(\boldsymbol{x}^*),\ \boldsymbol{\lambda}^* \ge \mathbf{0} \tag{32}$$

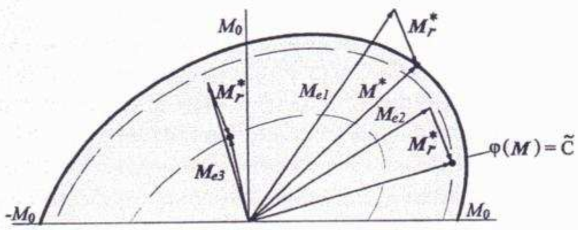

Fig 6. Huber-Mises yield conditions

resulted in the following vector of plastic multipliers $\boldsymbol{\lambda}^{*\,v=1}$: $\lambda_1^{*\,v=1} = 0.1064$, $\lambda_2^{*\,v=1} = 0.6300$ (other components equal to zero). The second problem aims at verifying the plate stiffness conditions $0 \le u_{r1} \le u_{r1,max}$. The residual deflection magnitude $u_{r1} = 0.2985\ M_0 R^2$ /K is obtained from the vector $\boldsymbol{u}_r^{*v=1} = \boldsymbol{H}\,\boldsymbol{\lambda}^{*v=1}$. The stiffness condition is satisfied but one must take into account the possible unloading of the cross-section. Therefore, the stiffness constraint $0 \le u_{r1} \le u_{r1,max}$ is changed to the stricter one (28): $0 \le u_{r1,sup} \le 0.84$. The upper bound of the deflection $u_{r1,sup}$ is calculated applying the mathematical model (29)-(31). Finally, the main problem (24)-(28) optimal solution reads: $q_{sup}^* = 5.7716\,M_0 R^{-2}$, $M_{sup}^* = 0.8091\ M_0$.

8 Conclusions

New potential, which is provided by connections between mathematical programming and extremum energy principles, are shown for formulation of analysis and optimization problems of shakedown theory and their numerical solution. Constructed mathematical models are universal: when stiffness constraints are neglected optimal solution is obtained according to cyclic–plastic failure, it is very easy to interpret monotonically increasing loading. Type of cyclic–plastic collapse is identified using complementary slackness conditions.

9 References

[1] Kaneko L, Maier G. Optimum design of plastic structures under displacement's constraints. Computer Methods in Applied Mechanics and Engineering, Vol 27 (3), p. 369–392, 1981

[2] Stein E, Zhang G, Mahnken R, Shakedown analysis for perfectly plastic and kinematic hardening materials.In: CISM. Progress in Computernal Analysis or Inelastic Structures. Wien, New York: Springer Werlag, p. 175–244, 1993

[3] Giambanco F, Palizzolo L, Polizzotto, C. Optimal shakedown design of beam structures. Structural Optimization, Vol 8, p. 156–167, 1994

[4] Tin–Loi F. Optimum shakedown design under residual displacement constraints. Structural and Multidisciplinary Optimization, Vol 19 (2), p. 130–139, 2000

[5] Kaliszky S, Lógó J. Plastic behaviour and stability constraints in the shakedown analysis and optimal design of trusses. Structural and Multidisciplinary Optimization, Vol 24 (2), p. 118–124, 2002

[6] Choi SH, Kim SE. Optimal design of steel frame using practical nonlinear inelastic analysis. Engineering Structures, Vol 24 (9), p. 1189–1201, 2002

[7] Staat M, Heitzer M (eds). Numerical methods for limit and shakedown analysis. Series of John von Neumann Institute for Computing, Vol 15, 2003

[8] Atkočiūnas J, Jarmolajeva E, Merkevičiūtė D. Optimal shakedown loading for circular plates. Structural and Multidisciplinary Optimization, Vol 27 (3), p. 178–188, 2004

[9] E. Raue, S. Hahn. Optimum reinforcement design of concrete cross-sections considering deformation constraints. Civil Engineering and Management, Vol XI (1), p. 65-71, 2005

[10] Atkočiūnas J, Borkowski A, König JA. Improved bounds for displacements at shakedown. Computer Methods in Applied Mechanics and Engineering, Vol 28 (3), p. 365–376, 1981

[11] Dorosz S, König JA. An iterative method of evaluation of elastic–plastic deflections of hyperstatic framed structures. Ingenieur–Archiv, Vol 55, p. 202–212, 1985

[12] Maier G, Comi C, Corigliano A, Perego U, Hübel H. Bounds and estimates on inelastic deformations: a study of their practical usefulness. European Commission Report, Nuclear Science and Technology Series, Brussels: European Commission, 1996

[13] Hachemi A, Weichert D. Application of shakedown theory to damaging inelastic material under mechanical and thermal loads. International Journal of Mechanical Sciences, Vol 39 (9), p. 1067–1076, 1997

[14] Lange–Hansen P. Comparative study of upper bound methods for the calculation of residual deformation after shakedown, Series R, No. 49. Lyngby: Technical University of Denmark, Dept. of Structural Engineering and Materials, 1998

[15] Bazaraa MS, Sherali HD, Shetty CM. Nonlinear programming: theory and algorithms. New York: Brijbasi Art Press Ltd., John Wiley & Sons, Inc., 2004

[16] Gutkowski W, Bauer J, Ivanov Z. Explicit formulation of Kuhn–Tucker necessary conditions in structural optimization. Computer and Structures, Vol 37 (5), p. 753–758, 1990

[17] Koiter, W. T., General theorems for elastic-plastic solids, Progress in Solid Mechanics, Sheddon, I. N. and Hills, R. Eds., North Holland, Amsterdam, p. 165-221, 1960.

[18] Kalanta S, Grigusevičius A. Formulation of framed structures equations by static and mixed methods. Civil Engineering and Management, IX, Suppl. 2, p. 100-112, 2003

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Paper 159

©Civil-Comp Press, 2008
Proceedings of the Ninth International Conference
on Computational Structures Technology,
B.H.V. Topping and M. Papadrakakis, (Editors),
Civil-Comp Press, Stirlingshire, Scotland

Optimal Shakedown Design of Frames Under Stability Conditions

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Abstract

A shakedown frames volume minimization and load optimization nonlinear mathematical models with strength, stiffness and stability constraints are investigated. There were developed methodology and algorithms for stability evaluation according to various design codes (Eurocode 3 (EC3) and Dutch NEN 6771) by integrating commercial software for the building industry MatrixFrame and the authors created nonlinear mathematical programming software. For the other investigators it provides the possibility to integrate the solutions of nonlinear programming problems (variables of plastic state: residual forces and displacements) into their structural design software. It is noteworthy, that proposed methodology allows the load combinations, occurring in the engineering practise realise as separate cases of variable repeated load. Numerical examples concerning optimization of frame structures are presented.

Keywords: optimal shakedown design, frames, stability, energy principles, mathematical programming.

1 Introduction

There are investigated the aspects of optimal shakedown design of bar structures under strength and stiffness conditions in details [1] - [8], although today the evaluation of stability conditions for the optimization problems of elastic-plastic frames remains topical scientific problem. For example, it is allowed to design elastic-plastic frames by EC3 or NEN 6771, but therein the methodology and algorithms for stability evaluation of shakedown structures are not fully elaborated. This had an influence on the topic of this paper: optimal shakedown design of frames, subjected to variable repeated load, under strength, stiffness and stability constraints. Herein two types of problems can be considered [9]. The first problem is optimal shakedown design of cross-sectional parameters (design problem) and the second one - load optimization problem for a frame subjected to variable repeated load. By solving load optimization problem maximal load variation bounds, ensuring adapted state of the frame and satisfying stiffness and stability requirements of the structure, are to be found.

Solution of frame optimization problems at shakedown is complicated as stress–strain state of dissipative systems depends on loading history [10]-[14]. These difficult optimization problems are implemented applying extremum energy principles and the theory of mathematical programming [15]. That enables to create new iterative algorithm based on Rosen project gradient method [16] - [17]. Evaluation of stability requirements for both optimization problems is implemented by integrating commercial software for the building industry MatrixFrame and the authors created nonlinear mathematical programming software. Numerical examples of the frames are presented. The results are valid for small displacement assumptions

2 General mathematical models

General mathematical models presented in Table 1 are the basis for the development of optimization mathematical models of frames at shakedown considered in this paper.

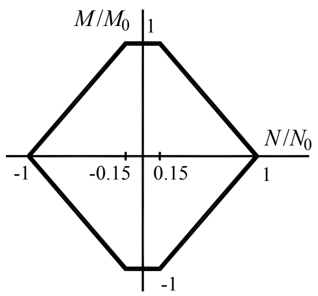
Volume minimization problem		Load optimization problem	
find		find	
$min\ \left(\boldsymbol{L}^T \boldsymbol{S}_0 - \boldsymbol{\lambda}_j^T \boldsymbol{\varphi}_j\right)$	(1)	$max\ \left(\boldsymbol{T}_{sup}^T \boldsymbol{F}_{sup} - \boldsymbol{T}_{inf}^T \boldsymbol{F}_{inf} - \boldsymbol{\lambda}_j^T \boldsymbol{\varphi}_j\right)$	(6)
subject to		subject to	
$\boldsymbol{\varphi}_j = \boldsymbol{S}_0 - \boldsymbol{\Phi}\left(\boldsymbol{G\lambda} + \boldsymbol{S}_{ej} + \boldsymbol{S}_{ec}\right) \geq \boldsymbol{0}$	(2)	$\boldsymbol{\varphi}_j = \boldsymbol{S}_0 - \boldsymbol{\Phi}\left(\boldsymbol{G\lambda} + \boldsymbol{S}_{ej} + \boldsymbol{S}_{ec}\right) \geq \boldsymbol{0}$	(7)
$\boldsymbol{\lambda}_j \geq \boldsymbol{0},\ \boldsymbol{\lambda} = \sum_j \boldsymbol{\lambda}_j,\ j \in J$	(3)	$\boldsymbol{\lambda}_j \geq \boldsymbol{0},\ \boldsymbol{\lambda} = \sum_j \boldsymbol{\lambda}_j,\ j \in J$	(8)
$\boldsymbol{S}_{min} \leq \boldsymbol{S}_0 \leq \boldsymbol{S}_{max}$	(4)	$\boldsymbol{0} \leq \boldsymbol{F}_{sup} \leq \boldsymbol{F}_{max},\ \boldsymbol{F}_{min} \leq \boldsymbol{F}_{inf} \leq \boldsymbol{0}$	(9)
$\boldsymbol{u}_{min} \leq (\boldsymbol{u}_r + \boldsymbol{u}_{ej} + \boldsymbol{u}_{ec}) \leq \boldsymbol{u}_{max}$	(5)	$\boldsymbol{u}_{min} \leq (\boldsymbol{u}_r + \boldsymbol{u}_{ej} + \boldsymbol{u}_{ec}) \leq \boldsymbol{u}_{max}$	(10)

Table 1: General mathematical models of optimization problems

In both volume minimization and load optimization problems objective functions are described by formulas (1) and (6), where the vectors $\boldsymbol{L}$, $\boldsymbol{T}_{sup}$ and $\boldsymbol{T}_{inf}$ contain coefficients of weight, $\boldsymbol{\lambda}_j^T \boldsymbol{\varphi}_j$ is the complementary slackness conditions of mathematical programming. Yield conditions $\boldsymbol{\varphi}_j$ ($j \in J$) are shown in formulas (2) and (7) , where j is the number of all possible combinations $\boldsymbol{F}_j$ of load bounds $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$. Formulas (4) and (9) are constraints for the problem unknowns. Vectors $\boldsymbol{S}_{max}$, $\boldsymbol{S}_{min}$, $\boldsymbol{F}_{max}$ and $\boldsymbol{F}_{min}$ play major role for stability evaluation. About this role see in Section 3. Stiffness constraints are shown in (5) and (10). Discrete model of the frame at shakedown consists of s ($k = 1,2,\ldots,s$, $k \in K$) finite elements. Limit force S_{0k} ($k \in K$) is assumed as constant in the whole finite element. The degree of freedom is m, corresponding m - vector of displacements - $\boldsymbol{u}_e = \left(u_{e,1}, u_{e,2}, \ldots, u_{e,m} \right)^T$. Nodal internal forces of the element compound one n – vector of discrete model forces $\boldsymbol{S} = \left(\boldsymbol{S}_1, \boldsymbol{S}_2, \ldots, \boldsymbol{S}_v, \ldots, \boldsymbol{S}_\zeta \right)^T = \left(S_z \right)^T$ and strains – n–vector $\boldsymbol{\Theta} = \left(\boldsymbol{\Theta}_1, \boldsymbol{\Theta}_2, \ldots, \boldsymbol{\Theta}_v, \ldots, \boldsymbol{\Theta}_\zeta \right)^T = \left(\boldsymbol{\Theta}_z \right)^T$, $v = 1, 2, \ldots, \zeta$ ($v \in Z$), $z = 1, 2, \ldots, n$. The total number of design sections is ζ.

Load $\boldsymbol{F}(t)$ is characterized by time t, independent variation bounds $\boldsymbol{F}_{sup} = \left(F_{1,sup}, F_{2,sup}, \ldots, F_{m,sup} \right)^T$ and $\boldsymbol{F}_{inf} = \left(F_{1,inf}, F_{2,inf}, \ldots, F_{m,inf} \right)^T$ ($\boldsymbol{F}_{inf} \le \boldsymbol{F}(t) \le \boldsymbol{F}_{sup}$). Elastic displacements $\boldsymbol{u}_e(t)$ and forces $\boldsymbol{S}_e(t)$ of the structure are determined using influence matrixes of displacements and forces, $\boldsymbol{\beta} = \left(\boldsymbol{AK} \boldsymbol{A}^T \right)^{-1}$, $\boldsymbol{\alpha} = \boldsymbol{KA}^T \boldsymbol{\beta}$, respectively: $\boldsymbol{u}_e(t) = \boldsymbol{\beta} \boldsymbol{F}(t)$, $\boldsymbol{S}_e(t) = \boldsymbol{\alpha} \boldsymbol{F}(t)$, $\boldsymbol{K} = \boldsymbol{D}^{-1}$. Here $\boldsymbol{A}$ is a coefficient matrix of equilibrium equations $\boldsymbol{A} \boldsymbol{S} = \boldsymbol{F}$ and $\boldsymbol{D}$ is a quasi-diagonal flexibility matrix. Residual displacements $\boldsymbol{u}_r$ and forces $\boldsymbol{S}_r$ are related to the vector of plasticity multipliers $\boldsymbol{\lambda}$ by influence matrixes $\boldsymbol{H}$ and $\boldsymbol{G}$: $\boldsymbol{u}_r = \overline{\boldsymbol{H}} \boldsymbol{\Phi}^T \boldsymbol{\lambda} = \boldsymbol{H} \boldsymbol{\lambda}$, $\boldsymbol{S}_r = \overline{\boldsymbol{G}} \boldsymbol{\Phi}^T \boldsymbol{\lambda} = \boldsymbol{G} \boldsymbol{\lambda}$, $\overline{\boldsymbol{H}} = \left(\boldsymbol{AKA}^T \right)^{-1} \boldsymbol{AK}$ and $\overline{\boldsymbol{G}} = \boldsymbol{KA}^T \overline{\boldsymbol{H}} - \boldsymbol{K}$. Here $\boldsymbol{\Phi}$ – the matrix of peace–wise linearized yield conditions $\boldsymbol{\varphi}_j$ (2) and (7). The number of all possible combinations $\boldsymbol{F}_j$ of load bounds $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$ is $p = 2^m$ ($\boldsymbol{F}_{inf} \le \boldsymbol{F}_j \le \boldsymbol{F}_{sup}$): $\boldsymbol{S}_{ej} = \boldsymbol{\alpha} \boldsymbol{F}_j$, $\boldsymbol{u}_{ej} = \boldsymbol{\beta} \boldsymbol{F}_j$, $j = 1, 2, \ldots, p$, ($j \in J$). It is possible directly evaluate not only variable repeated load $\boldsymbol{F}_j$ but also other loads $\boldsymbol{F}_c$ (for example persistent load) additionally including them into set J. Elastic forces $\boldsymbol{S}_{ec}$, and elastic displacements $\boldsymbol{u}_{ec}$ resulted by loads $\boldsymbol{F}_c$ are calculated by formulas $\boldsymbol{S}_{ec} = \boldsymbol{\alpha} \boldsymbol{F}_c$, $\boldsymbol{u}_{ec} = \boldsymbol{\beta} \boldsymbol{F}_c$.

Design of the frame for optimal parameters by mathematical model (1)–(5) is performed when yield limit σ_{yk} of the frame material and lengths L_k of its all elements k ($k \in K$) and load variation bounds $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$ are known. Depending on the cross-sectional shape various yield conditions can be considered. In this paper, the focus is placed on yield conditions for rolled I steel sections (Fig. 1).



$$\boldsymbol{\Phi}_v = \begin{bmatrix} 0.85 & c_k \\ -0.85 & c_k \\ 0.85 & -c_k \\ -0.85 & -c_k \\ 1 & 0 \\ -1 & 0 \end{bmatrix}, \; v \in Z$$

Figure 1: Linear yield conditions

Relation $c_k = \dfrac{M_{0k}}{N_{0k}}$, $k \in K$ should be prescribed in advance. Limit moment $M_{0k} = \sigma_{yk} W_{pl,k} = \xi(\sigma_{yk}, A_k)$ and limit axial force $N_{0k} = \sigma_{yk} A_k$ of the element are functions of cross-sectional area A_k and yield limit of material σ_{yk}. True, usually one or the other specific dimension of the cross-section (for instance, flange thickness t_f and web thickness t_w of I-section while the width of flange b and height h are fixed; see Examples 1 and 2) participate in functional relation $M_{0k} = \xi(\sigma_{yk}, A_k)$ instead of cross-sectional area A_k. Limit moments M_{0k} of the frame elements and vectors of plasticity multipliers $\boldsymbol{\lambda}_j \geq \mathbf{0}$, $j \in J$ are unknowns of nonlinear mathematical programming problem (1)-(5). Constructive requirements of frames $\boldsymbol{S}_{min}$ and $\boldsymbol{S}_{max}$ are shown in conditions (4). Problem (1)–(5) is not exactly the volume minimization problem, because limit moments M_{0k} are used in objective function. Limit moments $\boldsymbol{M}_0$ and influence matrixes $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{G}$, $\boldsymbol{H}$ are related with unknowns A_k, $k \in K$; the listed matrixes are recalculated during solution of the problem (1)–(5). If stiffness and stability constrains are neglected, cyclic-plastic collapse of the frame is reached. Optimal solution of the problem (1)–(5) is vectors $\boldsymbol{S}_0^*$ and $\boldsymbol{\lambda}_j^*$, $j \in J$.

In the case of variable repeated load, the problem of load variation bound (6)–(10) $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$ determination is important also. It stated as follows: find shakedown load variation bounds $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$, satisfying the prescribed optimality criterion $max\left(\boldsymbol{T}_{sup}^T \boldsymbol{F}_{sup} - \boldsymbol{T}_{inf}^T \boldsymbol{F}_{inf} - \boldsymbol{\lambda}_j^T \boldsymbol{\varphi}_j\right)$, also strength, stiffness and stability requirements of the structure. Here $\boldsymbol{T}_{sup}$, $\boldsymbol{T}_{inf}$ are the optimality criterion weight coefficient vectors. The vector of limit bending moments $\boldsymbol{M}_0$ and the limits $\boldsymbol{u}_{min}$, $\boldsymbol{u}_{max}$ of total displacements $\boldsymbol{u} = \boldsymbol{u}_r + \boldsymbol{u}_{ej} + \boldsymbol{u}_{ec}$ are known in the problem (6)–(10). Optimal solution of the problem (6)–(10) is vectors $\boldsymbol{F}_{sup}^*$, $\boldsymbol{F}_{inf}^*$ and $\boldsymbol{\lambda}_j^*$, $j \in J$.

2 Stability evaluation

Stability in the mathematical models (1)–(5) and (6)–(10) are evaluated through the constructive restrictions (4) and (9) respectively, which are calculated by stability requirements of design codes EC3 or NEN 6771 (or even other code). Various design codes are implemented in commercial software that is available for needs of designers. Authors of the paper for stability evaluation use software for building industry MatrixFrame, version 4.1. Stability check in MatrixFrame is performed for both mentioned design codes. In case of EC3 there are calculated buckling resistance of members according to formulas of design code: 6.46, 6.54, 6.62. In case of NEN 6771 stability check is performed by formulas: 12.2-3 and 12.3-2. Element k meet the requirements of stability when maximal stability unity check (UC_k) calculated

by formulas of design code is less or equal to unity. UC is the ratio of design value and design resistance.
The frame volume minimization is performed according to the mathematical models (1)–(5) by iterations:

Step 1. Influence matrixes $\boldsymbol{\alpha}^0$, $\boldsymbol{\beta}^0$, $\boldsymbol{G}^0$, $\boldsymbol{H}^0$, coefficients c_k^0, $k \in K$ of yield conditions are determined for the assumed initial cross-sectional areas A_k^0, $k \in K$. Constraints (4) for problem variables M_{0k} are neglected.

Step 2. Problem (1)–(5) is solved and the new distribution of limit moments M_{0k}^*, $k \in K$, is found. Selection of new sections can be performed by two ways: by changing cross-sectional dimension (continuous optimization) or by selecting them from available assortment of manufactured cross-sections by applying the formula $W_{pl}^* \geq M_{0k}^* / \sigma_{yk}$ (discrete optimization).

Step 3. Variables of plastic state, residual forces $\boldsymbol{S}_r$ and displacements $\boldsymbol{u}_r$, are introduced into MatrixFrame stability calculation. If the maximal stability $UC_k > 1$, $k \in K$, then by changing cross-sectional dimension or selecting from assortment is found cross-section heaving the property $UC_k \leq 1$. In this case $M_{0k,min}$ is found. This means so in next iteration limit moment M_{0k} should be greater or equal to $M_{0k,min}$.

Step 4. New influence matrixes $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{G}$, $\boldsymbol{H}$, coefficients c_k, $k \in K$ are determined for cross-sections with areas A_k obtained in Step 2.

Step 5. Problem (1)–(5) is solved again using recalculated matrixes $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{G}$, $\boldsymbol{H}$, coefficients c_k and $M_{0k,min}$ obtained in Step 3.

Step 6. Steps 3-5 are repeated until the cross-sectional areas A_k obtained in two consecutive steps do not differ.

Stability requirements for all elements k, $k \in K$ is evaluated in Step 3 by founding such cross-sections A_k ($M_{0k,min}$) that satisfies requirements $UC_k \leq 1$.
The frame load optimization is performed according to the mathematical models (6)–(10) by iterations too:

Step 1. Problem (6)–(10) is solved and the new distribution of load variation bound $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$ is found. Constraints (9) for problem variables $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$ are neglected.

Step 2. Variables of plastic state, residual forces $\boldsymbol{S}_r$ and displacements $\boldsymbol{u}_r$, are introduced into MatrixFrame stability calculation. If the maximal stability $UC_k > 1$, $k \in K$, then by changing load domain $\boldsymbol{F}_j$ is found such load domain that ensure $UC_k \leq 1$. In this case $\boldsymbol{F}_{max}$ and $\boldsymbol{F}_{min}$ are found. This means so in next iteration load variation bounds $\boldsymbol{F}_{sup}$ and $\boldsymbol{F}_{inf}$ can't exceed load variation bounds $\boldsymbol{F}_{max}$ and $\boldsymbol{F}_{min}$ satisfying requirements of stability.

Step 3. Problem (6)–(10) is solved again using load variation bounds $\boldsymbol{F}_{max}$ and $\boldsymbol{F}_{min}$ obtained in Step 2.

Step 4. Steps 2 and 3 are repeated until the load variations bounds $\boldsymbol{F}_{sup}$ and $\boldsymbol{F}_{inf}$ obtained in two consecutive steps do not differ.

Stability requirements for all elements k, $k \in K$ is evaluated in Step 2 by founding load variations bounds $\boldsymbol{F}_{max}$ and $\boldsymbol{F}_{min}$ that satisfies requirements $UC_k \leq 1$.

3 Numerical examples

3.1 Example 1

Proposed calculation technique is illustrated by example of volume minimization problem (1)-(5) of two-storey frame (Fig. 2) The software M0opt1, which is created by authors, is based on Rosen project gradient method [17] and applied for solution of presented numerical example. For stability evaluation is used MatrixFrame. Stability constraints are calculated according to design code EC3.

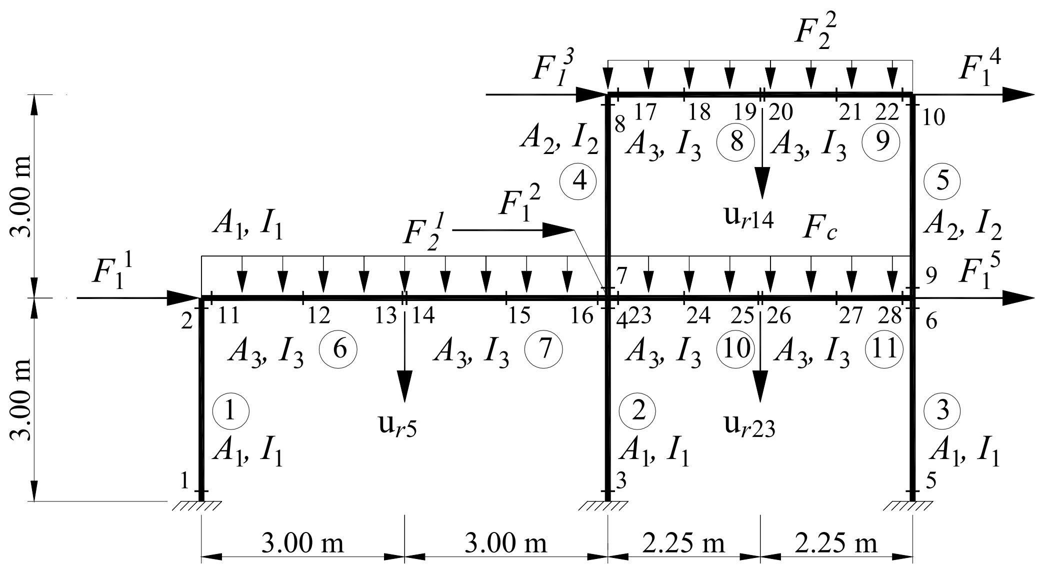


Figure 2: Discretized frame

The frame is subjected to two independent load sets: horizontal concentrated forces $\boldsymbol{F}_1 = \{F_1^1, F_1^2, F_1^3, F_1^4, F_1^5\}$ acting on the nodes of the frame and vertical uniformly distributed forces $\boldsymbol{F}_2 = \{F_2^1, F_2^2\}$ acting on the roof beams (6, 7, 8, 9), respectively. Permanent load $\boldsymbol{F}_c$ act on the floor beams (10, 11). Limits for the

variations of the load are defined by the inequalities $\mathbf{F}_{1,inf} \leq \mathbf{F}_1 \leq \mathbf{F}_{1,sup}$, $\mathbf{F}_{2,inf} \leq \mathbf{F}_2 \leq \mathbf{F}_{2,sup}$, where $\boldsymbol{F}_{1,inf} = \{-9.75, -4.9, -5, -6.75, -19.5\} \cdot \text{kN}$, $\boldsymbol{F}_{1,sup} = \{13, 6.5, 6.75, 5, 14.6\} \cdot \text{kN}$, $\boldsymbol{F}_{2,inf} = \{0, 0\}$, $\boldsymbol{F}_{2,sup} = \{48, 48\} \cdot \text{kN/m}$ and $\boldsymbol{F}_c = 117 \cdot$ kN/m.
The frame is made of steel with a modulus of elasticity $E = 210$ GPa and a yield limit $\sigma_y = 235$ MPa. The cross-sections of the frame columns, roof and floor beams are shown in Fig. 3.

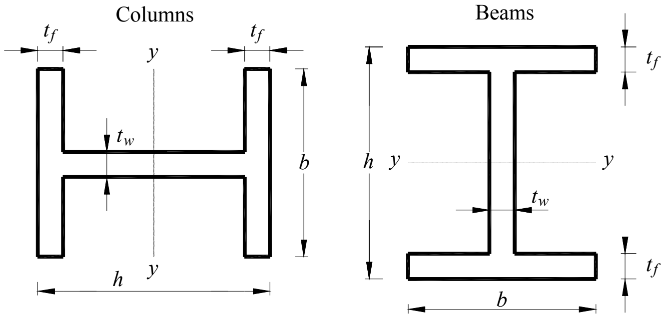


Figure 3: Cross-sectional shapes for frame columns and beams

The parameters b and h remain the same throughout the continues optimization process, only the thickness $t = t_f = t_w$ of the flanges and web varying. The values b and h of cross-sections are shown in Table 2. In case of discrete optimization cross-sections are selected from available assortment of manufactured cross-sections.

Elements k, $k \in K$	b [m]	h [m]
1, 2, 3	0.15	0.15
4, 5	0.1	0.12
6, 7, 8, 9, 10, 11	0.15	0.2

Table 2: Values of cross-sections

The limit forces of the cross-sections when $t = t_f = t_w$ are calculated according to

$$M_0 = \sigma_y W_{pl,y} = \sigma_y \left(t^3 - (b+h)t^2 + \left(\frac{h^2}{4} + bh \right) t \right), \; N_0 = \sigma_y A = \sigma_y \left(2bt + t(h-2t) \right).$$

The main task is to determine the minimum volume of the adapted frame (Fig. 2) in the case when the vector of inner forces of the discretized frame is $\boldsymbol{S} = (\boldsymbol{M}, \boldsymbol{N})^T$ $= (M_1, M_2, M_3, \ldots, M_{27}, N_1, N_2, \ldots, N_{11})^T = (S_i)^T$, $i = 1, 2, \ldots, n = 38$, i.e. both bending moments M and axial forces N are taken into account. In this case the frame volume minimization is performed according to the mathematical model (1)–(5). The unknowns are the cross-sectional areas of the frame columns and beams A_k, $k \in K$ and the vectors of plasticity multipliers $\boldsymbol{\lambda}_j$, $j = 1, 2, ..., 4$. Problem (1)–(5) was solved according to the sequence of operations shown in Section 2 and five calculation cases were investigated:

Case C1. When only strength constraints (2) are taken into account. Optimization continuous;
Case C2. When only strength (2) and stiffness (5) constraints are evaluated. The following total displacement constraints were imposed: $-\infty \leq u_5 \leq 0.03\,\text{m}$, $-\infty \leq u_{14} \leq 0.0225\,\text{m}$, $-\infty \leq u_{23} \leq 0.0225\,\text{m}$ (Fig. 2). Optimization continuous;
Case C3. When only strength (2) and constructive constraints (4) are taken into account. Optimization continuous;
Case C4. When only strength (2) and constructive constraints (4) are taken into account. Optimization discrete;
Case C5. When all (strength (2), stiffness (5) and constructive (stability) (4)) constraints are evaluated. The following total displacement constraints were imposed: $-\infty \leq u_5 \leq 0.03\,\text{m}$, $-\infty \leq u_{14} \leq 0.0225\,\text{m}$, $-\infty \leq u_{23} \leq 0.0225\,\text{m}$ (Fig. 2). Optimization continuous.

The calculation results for all described cases, depending on applied constraints, is shown in Table 3.

Case	M_{01} [Nm]	M_{02} [Nm]	M_{03} [Nm]	Objective function (OF)	Volume [m^3]	Location of the plastic strains
C1	75441	41673	204168	3991522	0.26149777	6, 2, 23
C2	93970	34942	223206	4403462	0.292369813	23
C3	120537	48302	186579	4173339	0.283231289	23
C4	174986	57610	189018	4755802	0.350856685	23
C5	108090	44151	215258	4466587	0.300776204	23

Table 3: Calculation results of volume minimization problem

In case of C2 and C5 total displacement u_{23} reach upper bound $u_{max} = 0.0225\,\text{m}$. When discrete optimization is applied for the case C4, limit moments $M_{01} = 174986\,\text{Nm}$, $M_{02} = 57610\,\text{Nm}$ and $M_{03} = 189018\,\text{Nm}$ correspond to the cross-sections HE240, HE160 and IPE330, respectively.
Convergence with desirable precision of the main optimization problem objective function is a criterion of the optimal solution. In the case C2 value of convergence $\delta = 0.25\%$, iteration process is shown in Table 4. Convergence of optimization problem objective function for all cases is shown in Figure 4.

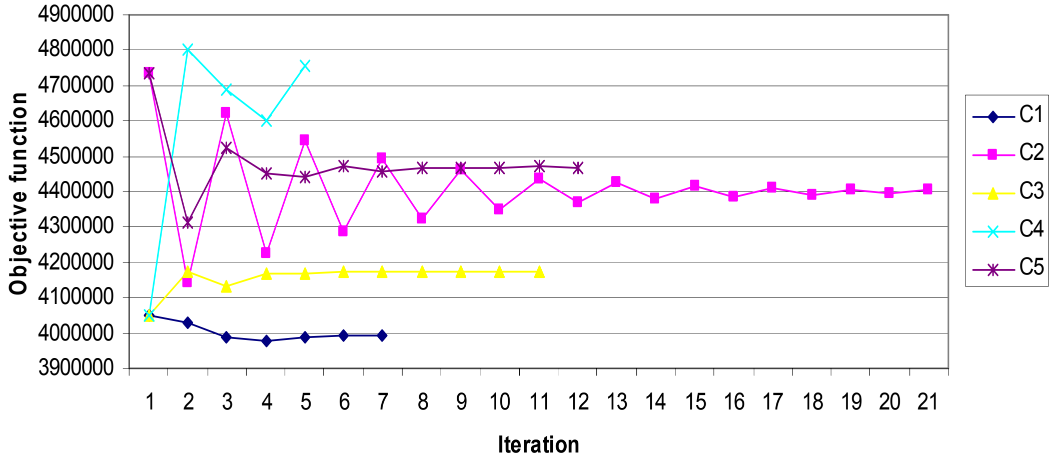


Figure 4: Convergence of optimization problem objective function

Iteration	M_{01} [Nm]	M_{02} [Nm]	M_{03} [Nm]	OF	δ OF %
1	96888	42400	240460	4733292	
2	93807	37591	204883	4143051	12,47
3	95221	37257	236064	4621487	-11,55
4	93755	35439	211158	4223807	8,61
5	94299	35814	231966	4543060	-7,56
6	93670	34931	215459	4284503	5,69
7	94140	35320	228876	4492323	-4,85
8	93767	34832	218090	4324254	3,74
9	94083	35129	226802	4459547	-3,13
10	93840	34837	219776	4350228	2,45
11	94044	35043	225444	4438312	-2,02
12	93885	34860	220870	4367176	1,60
13	94016	34999	224559	4424527	-1,31
14	93912	34882	221583	4378244	1,05
15	93997	34973	223983	4415558	-0,85
16	93929	34898	222047	4385447	0,68
17	93984	34958	223609	4409735	-0,55
18	93939	34909	222348	4390121	0,44
19	93975	34948	223365	4405942	-0,36
20	93946	34916	222545	4393195	0,29
21	93970	34942	223206	4403462	-0,23

Table 4: Convergence of optimization problem
objective function for case C2

3.2 Example 2

Proposed calculation technique is illustrated by example of load optimization problem (6)–(10) of two-storey frame (Fig. 2) The software MaxFopt1, which is created by authors, is based on Rosen project gradient method [17] and applied for solution of presented numerical example. For stability evaluation is used MatrixFrame. Stability constraints are calculated according to design code NEN 6771.

The frame is made of steel with a modulus of elasticity $E = 210\,\text{GPa}$ and a yield limit $\sigma_y = 235$ MPa. The cross-sections of the frame columns, roof and floor beams are shown in Fig. 3. Values of cross-section are shown in Table 4. Cross-sections remains not changed through entire optimization process.

Elements k, $k \in K$	b [m]	h [m]	t [m]	A_k [m^2]	M_{0k} [Nm]	N_{0k} [N]
1, 2, 3	0.15	0.15	0.016	0.006688	88665	1571680
4, 5	0.1	0.12	0.01	0.003000	31725	705000
6, 7, 8, 9, 10, 11	0.15	0.2	0.03	0.013200	21432	3102000

Table 4. Values of cross-sections

The frame is subjected to two independent load sets: horizontal concentrated forces $\boldsymbol{F}_1 = \{F_1^1, F_1^2, F_1^3, F_1^4, F_1^5\}$ acting on the nodes of the frame and vertical uniformly distributed forces $\boldsymbol{F}_2 = \{F_2^1, F_2^2\}$ acting on the roof beams (6, 7, 8, 9), respectively. Permanent load $\boldsymbol{F}_c = 117 \cdot$ kN/m act on the floor beams (10, 11). Limits for the variations of the load defined by the inequalities $\mathbf{F}_{1,inf} \le \mathbf{F}_1 \le \mathbf{F}_{1,sup}$, $\boldsymbol{F}_{2,inf} \le \boldsymbol{F}_2 \le \boldsymbol{F}_{2,sup}$, they are unknowns of the optimization problem. The main task is to determine the load variation bounds of the adapted frame (Fig. 2) in the case when the vector of inner forces of the discretized frame is $\boldsymbol{S} = (\boldsymbol{M}, \boldsymbol{N})^T$ $= (M_1, M_2, M_3, \ldots, M_{27}, N_1, N_2, \ldots, N_{11})^T = (S_i)^T$, $i = 1, 2, \ldots, n = 38$, i.e. both bending moments M and axial forces N are taken into account. In this case the frame load optimization is performed according to the mathematical model (6)–(10). The unknowns are the load variation bounds $\boldsymbol{F}_{1,inf}$, $\boldsymbol{F}_{2,inf}$, $\boldsymbol{F}_{1,sup}$ and $\boldsymbol{F}_{2,sup}$, and the vectors of plasticity multipliers $\boldsymbol{\lambda}_j$, $j = 1, 2, \ldots, 4$. Problem (6)–(10) was solved according to the sequence of operations shown in Section 2 and three calculation cases were investigated:

Case C1. When only strength constraints (7) are taken into account;

Case C2. When strength (7) and stiffness (10) constraints are taken into account. The following total displacement constraints were imposed: $-\infty \le u_5 \le 0.03\,\text{m}$, $-\infty \le u_{14} \le 0.0225\,\text{m}$, $-\infty \le u_{23} \le 0.0225\,\text{m}$ (Fig. 2).;

Case C3. When strength (7) and constructive constraints (9) are taken into account. The calculation results for all described cases, depending on applied constraints, is presented in Table 5.

Case	$\boldsymbol{F}_{1,sup}$	$\boldsymbol{F}_{2,sup}$	$\boldsymbol{F}_{1,inf}$	$\boldsymbol{F}_{2,inf}$	OF	Location of the plastic strains
C1	23679	44035	-29349	-10	97073	4, 6, 8, 23
C2	15777	26006	-23958	-10	65751	4, 6
C3	11839	19200	-14673	-10	45722	4

Table 5: Calculation results of load optimization problem

In case of C2 total displacement u_{23} reach upper bound $u_{max} = 0.0225\,\mathrm{m}$. Iterative solution process was performed only for case C3, while optimal solution for cases C1 and C2 were obtained in first iteration.

4 Conclusions

Practical implementation of the shakedown structures design methodology should be based not only on the theoretical improvements and created new mathematical models but also on close relation with existing building design. In this way it is possible to avoid the gap between the theoretical methods of structures optimization and real design that is based on design codes. For this purpose in this paper there are created main optimization problems with strength, stiffness and stability constraints where solution part that is related to stability is transferred to the design software with implemented design codes. Solution procedures become iterative: structural or load constraints of ordinary iteration of the main optimization problem are calculated with design software. On the other hand, initial data for design software become residual forces and residual displacements obtained from the solution of optimization problem i.e. influence of plastic deformations is evaluated. Convergence with desirable precision of the main optimization problem objective function is a criterion of the optimal solution. Proposed ways of optimization problems solution allow to realize discrete optimization principles. In such way shakedown theory become generalization tool for implementation of calculation and optimization for elastic-plastic structures in case of different loading.

References

[1] L. Kaneko, G. Maier, "Optimum design of plastic structures under displacement's constraints", Computer Methods in Applied Mechanics and Engineering, 27(3), 369–392, 1981.

[2] E. Stein, G. Zhang, R. Mahnken, "Shakedown analysis for perfectly plastic and kinematic hardening materials", In: CISM. Progress in Computernal Analysis or Inelastic Structures, Wien, New York, Springer Werlag, 175–244, 1993.

[3] F. Giambanco, L. Palizzolo, C. Polizzotto, “Optimal shakedown design of beam structures”, Structural Optimization, 8, 156–167, 1994.
[4] F. Tin–Loi, “Optimum shakedown design under residual displacement constraints”, Structural and Multidisciplinary Optimization, 19(2), 130–139, 2000.
[5] S. Kaliszky, J. Lógó, “Plastic behaviour and stability constraints in the shakedown analysis and optimal design of trusses”, Structural and Multidisciplinary Optimization, 24(2), 118–124, 2002.
[6] SH. Choi, SE. Kim, „Optimal design of steel frame using practical nonlinear inelastic analysis”, Engineering Structures, 24(9), 1189–1201, 2002.
[7] M. Staat, M. Heitzer, (eds), “Numerical methods for limit and shakedown analysis”, Series of John von Neumann Institute for Computing, 15, 306, 2003.
[8] S. Benfratello, L. Cirone, F. Giambanco, “A multicriterion design of steel frames with shakedown constraints”, Computers and Structures, 84, 269–282, 2006.
[9] A. A. Cyras, “Mathematical models for the analysis and optimization of elastoplastic structures”, Chichester, Ellis Horwood Lim., 121, 1983.
[10] J. Atkočiūnas, A. Borkowski, JA. König, “Improved bounds for displacements at shakedown”, Computer Methods in Applied Mechanics and Engineering, 28(3), 365–376, 1981.
[11] S. Dorosz, JA. König, ”An iterative method of evaluation of elastic–plastic deflections of hyperstatic framed structures”, Ingenieur–Archiv, 55, 202–212, 1985.
[12] G. Maier, C. Comi, A. Corigliano, U. Perego, H. Hübel, “Bounds and estimates on inelastic deformations: a study of their practical usefulness”, European Commission Report, Nuclear Science and Technology Series, Brussels, European Commission, 286, 1996.
[13] A. Hachemi, D. Weichert, “Application of shakedown theory to damaging inelastic material under mechanical and thermal loads”, International Journal of Mechanical Sciences, 39(9), 1067–1076, 1997.
[14] P. Lange–Hansen, “Comparative study of upper bound methods for the calculation of residual deformation after shakedown”, Series R, 49, Lyngby, Technical University of Denmark, Dept. of Structural Engineering and Materials, 74, 1998.
[15] D. Merkevičiūtė, J. Atkočiūnas, “Optimal shakedown design of metal structures under stiffness and stability constraints”, Journal of Constructional Steel Research, 62(12), 1270-1275, 2006.
[16] J. Atkočiūnas, D. Merkevičiūtė, A. Venskus, “Optimal shakedown design of bar systems: Strength, stiffness and stability constraints”, Computers & Structures, In Press, Corrected Proof.
[17] MS. Bazaraa, HD. Sherali, CM. Shetty, “Nonlinear programming: theory and algorithms”, New York, Brijbasi Art Press Ltd., John Wiley & Sons, Inc., 652, 2004.

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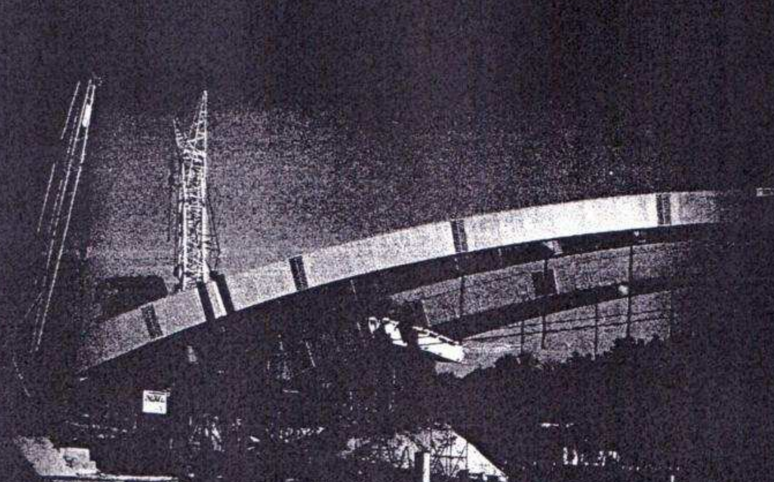
SELECTED PAPERS

Edited by
M. J. Skibniewski, P. Vainiūnas and E. K. Zavadskas



May 16–18, 2007 Vilnius, Lithuania

Vol III

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INTERNATIONAL ASSOCIATION FOR BRIDGE AND STRUCTURAL ENGINEERING

EUROPEAN COUNCIL OF CIVIL ENGINEERING

THE ASSOCIATION OF EUROPEAN CIVIL ENGINEERING FACULTIES

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VILNIUS GEDIMINAS
TECHNICAL UNIVERSITY

Organized by:
IABSE Lithuanian Group
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VGTU
LEIDYKLA
Vilnius TECHNIKA 2007

UDK 69(06)
Na269

The 9th International Conference Modern Building Materials, Structures and Techniques. Vol III. Selected papers of the 9th International Conference, held on May 16–18, 2007, Vilnius, Lithuania (Vilnius Gediminas Technical University, Lithuanian Academy of Science, International Association for Bridges and Structural Engineering, European Council of Civil Engineers, The Association of European Civil Engineering Faculties). Edited by M. J. Skibniewski, P. Vainiūnas and E. K. Zavadskas.Vilnius: Technika, 2007. 392 p.

The topics of reports are very diverse and include investigation and production of materials and structures, application of modern and effective calculation methods, optimization of structures and construction techniques, decision-making in construction, quality management, construction management and economics, advanced techniques for construction of buildings and civil engineering works, geotechnical problems, safety of people, ergonomics and fire protection for buildings and structures as well. At the conference researchers presented their reports in six sections: building materials and their technology; building technology and management; structural engineering and bridges; optimization of structures and new computation methods; geotechnics; fire protection and ergonomics.

All papers are reviewed

Knygos leidybą rėmė Lietuvos Respublikos švietimo ir mokslo ministerija
Supported by Ministry of Education and Science of Republic of Lithuania

http://leidykla.vgtu.lt
VGTU leidyklos TECHNIKA 1437 mokslo literatūros knyga

ISBN 978-9955-28-200-6 (Vol. 3)
ISBN 978-9955-28-201-3 (3 volumes)

The 9th International Conference
MODERN BUILDING MATERIALS, STRUCTURES AND TECHNIQUES. Vol I
Selected papers
(May 16–18, 2007, Vilnius, Lithuania)

2007-11-22. 49,0 sp. l. Tiražas 400 egz.
Vilniaus Gedimino technikos universiteto leidykla „Technika"
Saulėtekio al. 11, LT-10223 Vilnius, *http://leidykla.vgtu.lt*
Spausdino UAB „Baltijos kopija", Kareivių g. 13B, LT-09109 Vilnius, *www.kopija.lt*

CONTENTS

SESSION 4 – OPTIMIZATION OF STRUCTURES AND NEW COMPUTATION METHODS

May 16–18, 2007 m.
The 9th International Conference

MODERN BUILDING
MATERIALS, STRUCTURES
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DISCRETE OPTIMIZATION PROBLEMS OF THE STEEL BAR STRUCTURES

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Abstract. In this paper there are considered the optimal design problems of the elastic and elastic-plastic bar structures. These problems are formulated as nonlinear discrete optimization problems. The cross-sections of the bars are designed from steel rolled profiles. The mathematical models of the problems, including the structural requirements of the strength, stiffness and stability, are formulated in the terms of finite element method. The stated nonlinear optimization problems are solved by the iterative method, where each iteration comprises of the selection of the cross-sections of the bars from the assortment and solution of linear problems of the discrete programming. The requirement of discrete cross-sections is ensured by the branch and bound method.

Keywords: elastic and elastic-plastic steel bar structures, discrete optimization, finite element method. mathematical programming.

1. Introduction

For the purpose of saving material, the structures are designed by applying the methods of optimization [1-13]. The various algorithms for nonlinear optimization problems of structures are recently created: specific genetic [3-5], discrete optimization [6] and others optimization algorithms [7-11]. The solution algorithms for nonlinear optimization problems are not as universal as the latter for the linear problems. They are mostly dedicated to solve particular type of the problems. Furthermore, the problem of convergence of finding optimal solution occurs frequently, while they are applying. Therefore, nonlinear optimization problems frequently are solved by using the approximation technique when the linear programming problem is solved in each iteration. This method is applied in the paper [12], which is dedicated for the optimization of elastic structures. While designing the structures, an additional economy of the structural material is received for the structures with the plastic deformations in respect to optimal ones with the elastic deformations. However, the optimization problems of elastic-plastic structures [3, 8, 13] where are evaluated not only the strength, but also stiffness and stability requirements, are complex nonlinear programming problems and realization of them is complicated. In this paper design problems of the elastic and elastic-plastic steel structures are formulated as nonlinear optimization problems. Their mathematical models are created by using finite element method. In these models there are evaluated the conditions of strength, stiffness and stability [14]. The cross-sections are designed from standard steel rolled profiles. The formulated nonlinear optimization problems are solved by the iterative method where each iteration comprises of selection of the cross-sections of the bars from the assortment and solution of linear problems of the discrete programming. The requirement of discrete cross-sections is ensured by the branch and bound method.

2. The volume minimization problem for elastic structures

Mathematical models. There is considered the bar structure loaded by load combinations $v = 1,2,\ldots,p$, which bars designed from steel rolled profiles set $\mathbf{\Pi}$. Let the vector $\mathbf{A}_0$ denote the structural bars cross-sectional areas and $\mathbf{F}_v$, $\mathbf{S}_v$, $\mathbf{u}_v$ define the load, internal forces and displacements of v-th load combination, respectively. Then the volume (mass) minimization problem for the elastic structure is expressed by the following mathematical model:

find

$$\min f = \mathbf{L}^T \mathbf{A}_0 \tag{1}$$

subject to

$$[A]\mathbf{S}_v = \mathbf{F}_v, \ [\overline{D}]\mathbf{S}_v - [A]^T\mathbf{u}_v = \mathbf{0},$$
$$[G]\mathbf{A}_0 - [\overline{\Phi}]\mathbf{S}_v \geq \mathbf{0}, \ [E]\mathbf{u}_v \leq \mathbf{u}^+,$$
$$v = 1,2,...,p; \ \mathbf{A}_0 \geq \mathbf{A}_0^-, \ \mathbf{A}_0 \in \mathbf{\Pi}.$$

In this model: equalities – equilibrium and geometrical equations, describing the structural forces and displacements; first inequality – strength and stability conditions; other inequalities – displacements (stiffness) and constructive constraints. $\mathbf{L}$ is the vector of the structural elements lengths. The unknowns of this problem are the vectors $\mathbf{A}_0$, $\mathbf{S}_v$ and $\mathbf{u}_v$. Thus, the objective function of the problem expresses volume and the mass of the structure in the same time. Flexibility matrix $[\overline{D}]$ of the structural elements together with the strength and stability matrix $[\overline{\Phi}]$ depend on unknown $\mathbf{A}_0$. Therefore the model (1) is the nonlinear programming problem: the cross-sections of the structural bars, satisfying the requirements of the minimum volume (mass) of the structure, strength, stiffness and stability, are searching.

By eliminating the internal forces $\mathbf{S}_v = [\overline{D}]^{-1}[A]^T\mathbf{u}_v$ and geometrical equations, this model can be rewrote to the following optimization problem:

find

$$\min \ f = \mathbf{L}^T\mathbf{A}_0$$

subject to

$$[\overline{K}]\mathbf{u}_v = \mathbf{F}_v, \ [G]\mathbf{A}_0 - [\overline{\Phi}_u]\mathbf{u}_v \geq \mathbf{0}, \quad (2)$$
$$[E]\mathbf{u}_v \leq \mathbf{u}^+, \ v = 1,2,...,p;$$
$$\mathbf{A}_0 \geq \mathbf{A}_0^-, \ \mathbf{A}_0 \in \mathbf{\Pi};$$

here $[\overline{\Phi}_u] = [\overline{\Phi}][D]^{-1}[A]^T$; $[\overline{K}] = [A][\overline{D}]^{-1}[A]^T$ is the global stiffness matrix of the structure.

Formulation of the main relationships. The main dependencies composing the problems (1) and (2) are formulated in the terms of finite element method. For this purpose the structure is divided into the elements (bars) $k = 1,2,...,r$ joined in the nodes. The dependencies of the model (1) can be composed by using the equilibrium finite element method [15], and the model (2) can be created with the help of the equilibrium or geometrically compatible finite element method [16], because the stiffness matrix $[\overline{K}]$ can be formulated not only of the indicated formula, but also of the stiffness matrices of elements too.

Two equations groups compose *the equilibrium equations* $[A]\mathbf{S}_v = \mathbf{F}_v$:

1. *the equilibrium equations for nodes* describing the relation between the nodal forces of connected into nodes elements and the external forces acting on the nodes;
2. *the equilibrium equations for elements* describing the relation between the nodal forces and acting on the element external load, and are formulated only for elements affected by a distributed load. Expressions of these equations are presented in the papers [12, 15].

The equilibrium equation matrix $[A]$ could be formulated from the coefficients of the equilibrium equations of nodes and elements or from the formula $[A] = [C]^T[\overline{A}]$ [15]; here compatibility matrix $[C]$ describing relation between global displacements of the structural nodes and nodal displacements of elements; $[\overline{A}] = diag[A_k]$ is the quasi-diagonal matrix, which diagonal sub-matrices are composed from the coefficients of the static equations $\mathbf{P}_k = [A_k]\mathbf{S}_k$ of elements.

Flexibility matrix $[\overline{D}] = diag[\overline{D}_k]$ of *geometrical equations* $[\overline{D}]\mathbf{S}_v - [A]^T\mathbf{u}_v = \mathbf{0}$ contains in principal diagonal the flexibility matrices of finite elements $[\overline{D}_k]$. Its coefficients are calculated by formula $d_{ij} = d_k \int_{l_k} H_{ki}(x)H_{kj}(x)dx$, here $H_{ki}(x)$ is the form function of the internal forces; flexibility of the element under tension or compression is $d_k = 1/EA_k$, flexibility of an element under bending is $d_k = 1/EI_k$; E is the elasticity modulus, A_k, I_k are the cross-sectional area and moment of inertia, respectively. Expressions of a matrix $[\overline{D}_k]$ are given in the paper [17].

Strength and stability condition. Strength condition of the element under bending and tension or compression of j-th section is described via inequalities:

$$N_j + c_jM_j - RA_j \leq 0, \ -N_j + c_jM_j - RA_j \leq 0,$$
$$N_j - c_jM_j - RA_j \leq 0, \ -N_j - c_jM_j - RA_j \leq 0. \quad (3)$$

Here $R = f_{y,d}\gamma_c$; $f_{y,d}$ is the yield strength; γ_c is the partial factor of the exploitation conditions; $c_j = A_j / W_{ej}$; A_j, W_{ej} cross-sectional area and section modulus, respectively.

Furthermore, the bars under compression must satisfy the stability condition

$$-N_j / \varphi_j \leq RA_j \quad \text{or} \quad -N_j / \varphi_j - RA_j \leq 0. \quad (4)$$

The reduction factor φ for bars under centric or eccentric compression is defined in the national standard of civil engineering [14]. Strength conditions (3) create for all nodes of elements and stability conditions (4) only for the elements under compression. All of them are described via inequality $[G]\mathbf{A}_0 - [\overline{\Phi}]\mathbf{S}_v \geq \mathbf{0}$.

Solution algorithms. The direct solution of the nonlinear discrete programming problems (1) and (2) is fairly complicated. However, their solutions can be found in the iterative process, where in each iteration the cross-sectional profile is selected from the assortment and the linear programming problems solves, which obtain when matrices $[\overline{D}]$, $[\overline{\Phi}]$ and $[\overline{K}]$, $[\overline{\Phi}_u]$ of models (1) and (2) are replaced by matrices $[D]$, $[\varPhi]$ and $[K]$, $[\varPhi_u]$, which all coefficients are known, because the cross-sections of bars are set. The iterative process is finished, when it is received cross-sectional areas coincide with the previously set ones. For the purpose of minimizing of problem volume it is possible to consider each load case separately and for every one solve such problem:

find

$$\min\ f = \mathbf{L}^T \mathbf{A}_{0v}$$

subject to

$$[A]\mathbf{S}_v = \mathbf{F}_v,\ [D]\mathbf{S}_v - [A]^T \mathbf{u}_v = \mathbf{0};\qquad (5)$$
$$[\mathrm{G}]\mathbf{A}_{0v} - [\Phi]\mathbf{S}_v \geq \mathbf{0},\ [\mathrm{E}]\mathbf{u}_v \leq \mathbf{u}^+;$$
$$\mathbf{A}_{0v} \geq \mathbf{A}_{0,v-1},\ \ \mathbf{A}_{0v} \in \mathbf{\Pi}$$

or

find

$$\min\ f = \mathbf{L}^T \mathbf{A}_{0v}$$

subject to

$$[K]\mathbf{u}_v = \mathbf{F}_v;\qquad (6)$$
$$[\mathrm{G}]\mathbf{A}_{0v} - [\Phi]\mathbf{S}_v \geq \mathbf{0},\ [\mathrm{E}]\mathbf{u}_v \leq \mathbf{u}^+;$$
$$\mathbf{A}_{0v} \geq \mathbf{A}_{0,v-1},\ \ \mathbf{A}_{0v} \in \mathbf{\Pi}.$$

Inequality $\mathbf{A}_{0v} \geq \mathbf{A}_0^-$ for the load cases $v > 1$ is replaced by the condition $\mathbf{A}_{0v} \geq \mathbf{A}_{0,v-1}$. The vector $\mathbf{A}_{0p}$ corresponding to the last load case is the solution of the problems (1) and (2).

Furthermore, the optimization problems (5) and (6) can be solved in two stages:

1) classic problem of structural mechanics is solved i.e. the displacements $\mathbf{u}_v = [K]^{-1}\mathbf{F}_v$ and internal forces $\mathbf{S}_v = [D]^{-1}[A]^T\mathbf{u}_v$ are calculated; for this can be applied the equilibrium or geometrically compatible finite element method and various state-of-the-art computer technology dedicated for this kind of problems;

2) it is determining the vector of strength and stability conditions $\mathbf{S}_{0v} = [\varPhi]\mathbf{S}_v$ and solving the minimization problem:

find

$$\min\ f = \mathbf{L}^T \mathbf{A}_0$$

subject to

$$[G]\mathbf{A}_0 \geq \mathbf{S}_{0v},\ [G_0]\mathbf{A}_0 \geq [E]u_v,\qquad (7)$$
$$\mathbf{A}_0 \geq \mathbf{A}_0^-,\ \mathbf{A}_0 \in \mathbf{\Pi},\ v = 1, 2, ..., p.$$

Here unknown is the vector $\mathbf{A}_0$, whereas $\mathbf{S}_{0v} = [\varPhi]\mathbf{S}_v$. Having software for the internal forces calculations, solution method is easier, because volume of this problem is smaller. It should be noted that it is possible to search for the optimal solution when stability requirements are neglected. But in this case it is necessary to verify if received cross-sections of bars under compression satisfy stability conditions. If they are violated, then cross-sections should be augmented and additional calculation iteration should be performed with including into the mathematical model stability conditions.

Example 1. Let the bar structure, shown in Fig 1, be loaded by three load cases: I – $p_1 = 16{,}4$ kN/m, $p_2 = 16{,}4$ kN/m; II – $p_1 = 16{,}4$ kN/m, $p_2 = 4$ kN/m; III – $p_1 = 4$ kN/m, $p_2 = 16{,}4$ kN/m. Moreover, the vertical load $F = 27{,}6$ kN and indicated wind load acts in each load case. The optimal cross-sections from steel rolled profiles must be found. Columns and the upper chord are designed from I profiles and others bars from the hollow rectangle tubes. Yield strength $R_y = 275$ MPa, elasticity module $E = 2{,}1 \cdot 10^5$ MPa. Stiffness requirements are described via constraints $u_x \leq 5$ cm and $u_y \leq 10$ cm, here u_x is the horizontal displacement of top node of the column; u_y is the vertical displacement in the middle of the bottom chord of the truss.

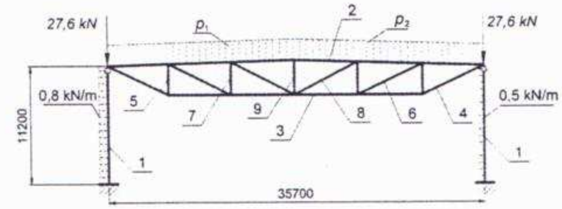

Fig 1. Calculations schema of the framed truss

The columns and the upper chord are calculated as the elements under bending and compression and other ones are calculated as the elements under tension or compression. Cross-sections are selected from the assortment. Initial height of the truss $h = 3{,}3$ m. After optimization it was obtained the following cross-sections: 1 – HEA300; 2 – IPE330; 3 – □ 180x180x6; 4 – □ 150x150x5; 5 – □ 90x90x5; 6 – □ 90x90x4; 7 – □ 70x70x4; 8 - □ 80x80x4; 9 – □ 60x60x5. Total weight of the optimal structure is 5229 kg.

Optimization of the structure is influenced not only by the height of the truss, but also by the web shape and the length of the segments. For this purpose the problems of truss height and web shape were created and considered.

3. Truss height and web shape optimization problems

In this section there are considering and formulating the optimal height and the rational shape of bottom chord of the framed truss, shown in Fig 1, search problems. Two designed versions are considering: 1) truss with horizontal bottom chord (Fig 1); 2) truss with parabolic bottom chord (Fig 2). Height optimization problems of theses trusses are described by such mathematical models:

a) truss with parabolic bottom chord

find

$$\min \ \mathbf{L}^T \mathbf{A}_0$$

subject to

$$[A(\mathbf{l})]\mathbf{S}_v = \mathbf{F}_v, \quad [D(\mathbf{l}, \mathbf{A}_0)]\mathbf{S}_v - [A(\mathbf{l})]^T \mathbf{u}_v = \mathbf{0},$$
$$[G]\mathbf{A}_0 - [\Phi(\mathbf{A}_0)]\mathbf{S}_v \geq \mathbf{0}, \quad [E]\mathbf{u}_v \leq \mathbf{u}^+,$$
$$v = 1, 2, \ldots, p;$$
$$l_j = \left[l_{jx}^2 + \left(y_{j2} - y_{j1}\right)^2\right]^{1/2}, \quad j = 1, 2, \ldots, s_1;$$
$$l_j = \left[l_{jx}^2 + \left(y_{j2} + y_{0j}\right)^2\right]^{1/2}, \quad j = 1, 2, \ldots, s_t;$$
$$y_{ji} - a_{ji} f = 0, \ \mathbf{A}_0 \geq \mathbf{A}_0^-, \ \mathbf{A}_0 \in \mathbf{\Pi};$$

b) truss with horizontal bottom chord

find

$$\min \ \mathbf{L}^T \mathbf{A}_0$$

subject to

$$[A(\mathbf{l})]\mathbf{S}_v = \mathbf{F}_v,$$
$$[D(\mathbf{l}, \mathbf{A}_0)]\mathbf{S}_v - [A(\mathbf{l})]^T \mathbf{u}_v = \mathbf{0},$$
$$[G]\mathbf{A}_0 - [\Phi(\mathbf{A}_0)]\mathbf{S}_v \geq \mathbf{0},$$
$$[E]\mathbf{u}_v \leq \mathbf{u}^+, \ \mathbf{A}_0 \geq \mathbf{A}_0^-, \ \mathbf{A}_0 \in \mathbf{\Pi},$$
$$l_j = \left[l_{jx}^2 + \left(f + y_{0j}\right)^2\right]^{1/2},$$
$$j = 1, 2, \ldots, s_t; \quad v = 1, 2, \ldots, p.$$

Here s_1 is number of bottom chord bars; s_t – number of web bars; f – height of the truss; l_j – length of j-th bar, $a_{ji} = 4x_i(l - x_i)/l^2$, l – length of the span. Main unknowns of these problems are cross-sectional areas A_j of bars and height of truss f. There are nonlinear programming problems, which can be solved iteratively.

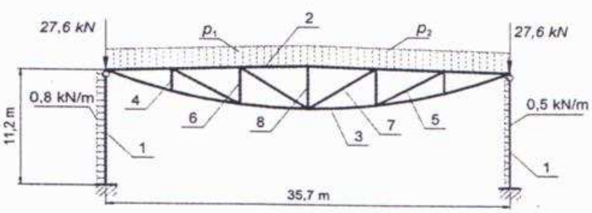

Fig 2. Framed truss with parabolic sketch bottom chord

Example 2. For the analyses of the framed structure in the first example, which is loaded by three prescribed load cases, must be determined: 1) truss rational bottom chord sketch; 2) rational length of the web segment and bars placing; 3) optimal height of the truss. The results of truss investigation are presented in the Figs 3-5. While performing the analysis of truss bottom chord sketch and web structure it were comparing weight of optimal frame with horizontal and parabolic bottom of truss, while number of truss segments was equal to 6, 8, and 10, and its height $h = 3{,}3 \div 4{,}5$ m. In the Fig 3 are shown the results of these investigations. The results of calculations showed, that more rational was the truss with horizontal bottom chord.

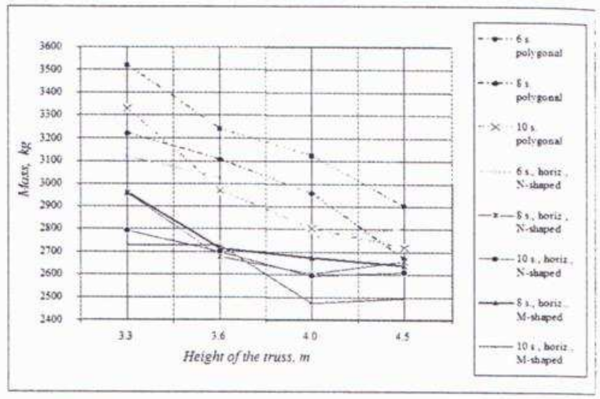

Fig 3. Analysis results of various web and chords shapes

It was investigated N-shaped truss (Fig 1) and M-shaped truss (Fig 4). It was determined, that most rational was the structure of the web which was showed in the Fig 4, and the optimal height of such truss was $h = 4$ m.

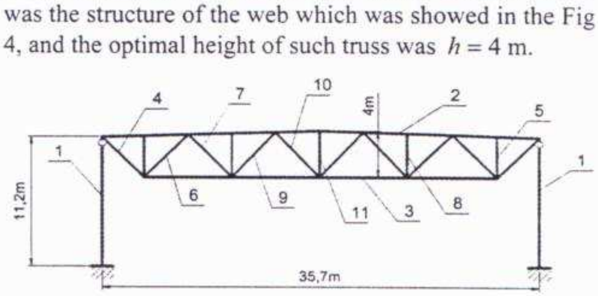

Fig 4. Framed truss with the optimal shape web

In the Fig 5 are showed chords, web and total mass of truss with optimal shape web dependence on its height. In the Fig 3 and Fig 5 are showed only the mass of trusses (mass, equal to 1982 kg, of the columns are not evaluated).

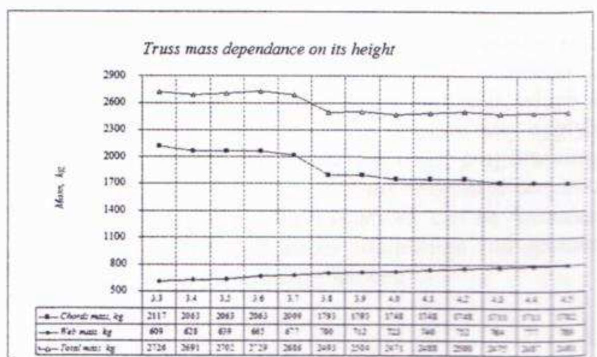

Fig 5. Investigations results of the optimal web truss height

4. The problem of elastic- plastic structure volume optimization

Mathematical models. In the case of the monotonically increasing load the mathematical model of the problem of the minimal volume (mass) elastic-plastic structure can be formulated according to the corresponding optimization model of elastic structure, when the plastic strains $\mathbf{\Theta}_p = [\Phi]^T \boldsymbol{\lambda}$ and additional complementary slackness condition are evaluated

$$\boldsymbol{\lambda}^T \left\{[G]\mathbf{A}_0 - [\overline{\Phi}]^T \mathbf{S}\right\} = 0 \tag{8}$$

that must correspond plastic multipliers $\boldsymbol{\lambda} \geq \mathbf{0}$. So, referring to the model (1), it is received such, monotonically increasing load acting on elastic-plastic structure, which corresponds to the requirements of the strength, stiffness and stability, mathematical model of optimization problem:

find

$$\min \mathbf{L}^T \mathbf{A}_0$$

subject to

$$\begin{gathered}[A]\mathbf{S} = \mathbf{F},\ [\overline{D}]\mathbf{S} + [\overline{\Phi}]^T \boldsymbol{\lambda} - [A]^T \mathbf{u} = \mathbf{0}, \\ \boldsymbol{\lambda}^T \left\{[G]\mathbf{A}_0 - [\overline{\Phi}]\mathbf{S}\right\} = 0,\ \boldsymbol{\lambda} \geq \mathbf{0},\ [E]\mathbf{u} \leq \mathbf{u}^+, \\ [G]\mathbf{A}_0 - [\overline{\Phi}]\mathbf{S} \geq \mathbf{0},\ \mathbf{A}_0 \geq \mathbf{A}_0^-,\ \mathbf{A}_0 \in \mathbf{\Pi}.\end{gathered} \tag{9}$$

The search of this nonlinear programming problem solution $\mathbf{S}, \mathbf{u}, \boldsymbol{\lambda}, \mathbf{A}_0$ is very difficult. It is especially hardened by the nonlinear conditions (8). That's why it is solved in iteration way, in each iteration selecting cross-sections of bars and solving simpler problem of nonlinear programming, in which only additional complementary slackness conditions are nonlinear. For the purpose of admissible (design) set simplification of the problem and its numerical realization, it is needed to eliminate these conditions from the constraints of the problem. This can be done in two ways - by moving them to the objective function (such possibility is proved in the paper [18] and used in the paper [19]) or eliminating and solving reduced optimization problem. So in each iteration it is possible to solve such problem:

find

$$\min f = \mathbf{L}^T \mathbf{A}_0 + \boldsymbol{\lambda}^T \{[G]\mathbf{A}_0 - [\Phi]\mathbf{S}\}$$

subject to

$$\begin{gathered}[A]\mathbf{S} = \mathbf{F},\ [D]\mathbf{S} + [\Phi]^T \boldsymbol{\lambda} - [A]^T \mathbf{u} = \mathbf{0}, \\ [G]\mathbf{A}_0 - [\Phi]\mathbf{S} \geq \mathbf{0},\ \boldsymbol{\lambda} \geq \mathbf{0},\ [E]\mathbf{u} \leq \mathbf{u}, \\ \mathbf{A}_0 \geq \mathbf{A}_0^-,\ \mathbf{A}_0 \in \mathbf{\Pi}\end{gathered} \tag{10}$$

or

find

$$\min f = \mathbf{L}^T \mathbf{A}_0$$

subject to

$$\begin{gathered}[A]\mathbf{S} = \mathbf{F},\ [G]\mathbf{A}_0 - [\Phi]\mathbf{S} \geq \mathbf{0}, \\ [D]\mathbf{S} + [\Phi]^T \boldsymbol{\lambda} - [A]^T \mathbf{u} = \mathbf{0},\ \boldsymbol{\lambda} \geq \mathbf{0}, \\ [E]\mathbf{u} \leq \mathbf{u}^+,\ \mathbf{A}_0 \geq \mathbf{A}_0^-,\ \mathbf{A}_0 \in \mathbf{\Pi}.\end{gathered} \tag{11}$$

In the first case it is received the problem with nonlinear objective function and liner constrains, and in the second case - the reduced linear programming problem (RLPP). It's understandable, that while solving RLPP, the condition $\lambda_j \left\{[G_j]A_0 - [\Phi_j]S\right\} = 0$ of some calculated section won't be satisfied. Therefore in this case for defining the optimal solution it is needed to apply the method of branch and bound, setting additional constraints $\lambda_j \leq 0$. for the recent sections.

Example 3. It is needed to set the cross-sections of the bars of the steel rolled profiles of the optimal framed structure, which calculations scheme is exampled in the Fig 1. The height of the truss is $h = 3{,}3$ m.

The columns and the upper chord of the truss are designed from I profiles, and other bars - from rectangle profile tube. The yield strength of the metal $R_y = 275$ MPa, elasticity module $E = 2{,}1 \cdot 10^5$ MPa. The requirements of the strength is described via constraints $u_x \leq 5$ cm and $u_y \leq 10$ cm; here u_x – horizontal displacement of columns top node, u_y – vertical displacement of truss bottom chord middle node.

Frame bars optimal cross-sections were determined with the help of the branch and bound method by solving reduced nonlinear programming problems. It were received such cross-sections of the bars: 1 – HEA300; 2 – IPE330; 3 – □ 180x180x6; 4 – □ 140x140x5; 5 – □ 90x90x5; 6 – □ 90x90x4; 7 – □ 70x70x4; 8 – □ 80x80x4; 9 – □ 60x60x5. This solution show, that while designing structure, in which it is allowed plastic deformations, it is possible to reduce only tension 4-th bar cross-section. Minimal mass of the optimal elastic-plastic structure $f = 5178$ kg is only 51 kg smaller than the mass of the optimal elastic structure.

5. Conclusions

1. The problems of the steel structures designing are formulated as nonlinear optimization problems. It is demonstrated, that elastic and elastic-plastic structures designing from rolled profiles problems are nonlinear discrete optimization problems, which solutions can be found in the iterative way applying branch and bound method and linear programming.
2. It is proposed three algorithms of optimal bars structures designing, which relations can be formulated applying the methods of equilibrium and geometrically compatible finite elements.

3. It was formulated the problem of truss optimal height determination and, performed calculations it was established, that most rational is 4 m height, i.e. $1/9 \cdot l$ truss (l - length of the span).
4. While performed analysis of the bottom chord sketch, as it were various height of the truss, it was determined that more rational is the truss with parallel bottom chord (Fig 1), comparing with the truss which bottom chord was form of quadratic parabola (Fig 2).
5. While fulfilling the analysis of the truss web form and density it was determined, that most rational is the triangle web with vertical bars (Fig 3), while length of segment is 3,6 m or $1/10 \cdot l$.
6. Elastic-plastic framed structure analysis confirmed statement, that often optimal structure project is determined not by the strength, but stiffness, stability and structural requirements.

References

1. BANICHUK, N. V. *Introduction to the optimization of the structures*. Moscow, 1986. 302 p. (in Russian).
2. MAJID, K. I. *Optimum design of structures*. Moscow, 1979. 237 p. (in Russian).
3. HAYALIOGLU, M. S. Optimum design of geometrically non-linear elastic-plastic steel frames via genetic algorithm. *Computers & Structures*, 77, 2000, p. 527–538.
4. HAYALIOGLU, M. S.; DEGERTEKIN, S. O. Design of non-linear steel frames for stress and displacement constraints with semirigid connections via genetic optimization. *Structural and Multidisciplinary Optimization*, 27, 2004, p. 259–271.
5. ZHENG, Q. Z.; QUERIN, O. M.; BARTON D. C. Geometry and sizing optimisation of discrete structure using the genetic programming method. *Structural and Multidisciplinary Optimization*, 31(6), 2006, p. 452–461.
6. GUTKOWSKI, W., editor. *Discrete Structural optimization*. Springer-Verlag, 1997. 250 p.
7. MANICKARAJAH, D.; XIE, Y. M.; STEVEN, G. P. Optimum design of frames with multiple constraints using an evolutionary method. *Computers & Structures*, 74, 2000, p. 731–741.
8. KARKAUSKAS, R. Optimization of elastic-plastic geometrically non-linear light-weight structures under stiffness and stability constraints. *Civil Engineering and Management*, 10(2), p. 97–106.
9. YUGE, K.; IWAI, N.; KIKUCHI, N. Optimization of 2D structures subjected to non-linear deformations using the homogenization method. *Structural optimization*, 17, 1999, p. 286–299.
10. FENG, F. Z.; KIM, Y. H.; YANG, B. S. Application of hybrid optimization techniques for model updating of rotor shafts. *Structural and Multidisciplinary Optimization*, 32(1), 2006, p. 67–75.
11. MERKEVIČIŪTĖ, D.; ATKOČIŪNAS, J. Optimal shakedown design of metal structures under stiffness and stability constraints. *Journal of Constructional Steel Research*, 62, 2006, p. 1270–1275.
12. JANULEVIČIUS, R.; KALANTA, S. Optimization of elastic beam structure using linear programming. In *Material of 8th conference of young Lithuanian scientist "Science - Future of Lithuania"*, held in Vilnius in March 24-25. Vilnius: Technika, 2005, p. 194–204. (in Lithuanian).
13. KALANTA, S.; GRIGUSEVIČIUS, A. Modeling of elastic-plastic beam structures optimization problems by finite element method. In *The 8th International conference Modern Building Materials, Structures and Techniques*, Selected papers. Ed by E. K. Zavadskas, P. Vainiūnas and F. M. Mazzolani. Vilnius: Technika, 2004, p. 782–789.
14. Design code STR2.05.08:2005. *Design of steel structures. Principal guidelines*. Vilnius, 2005. 128 p. (in Lithuanian).
15. KALANTA, S. The equilibrium finite element in computation of elastic structures. *Statyba*, Vilnius: Technika, 1(1), 1995, p. 25–47. (in Russian).
16. BARAUSKAS, R.; BELEVIČIUS, R.; KAČIANAUSKAS, R. *Basics of the finite element method*. Vilnius: Technika, 2004. 612 p. (in Lithuanian).
17. KALANTA, S. Calculation of the elastic plane bars structures using the finite element method. Questions of theoretical and applied mechanics. *Lithuanian Journal of Computational Mechanics*, 26, 1983, p. 78–94. (in Russian).
18. KALANTA, S. New formulations of optimization problems of elastoplastic bar structures under displacement constraints. *Mechanika*, Kaunas: Technologija, 5(20), 1999, p. 9–16. (in Russian).
19. VENSKUS, A.; ATKOČIŪNAS, J. Improved solution algorithm for shakedown optimization problems. In *Material of 9th conference of young Lithuanian scientist "Science - Future of Lithuania"*, held in Vilnius in March 29-31. Vilnius: Technika, 2006, p. 265–270. (in Lithuanian).


Modern Building Materials, Structures and Techniques
Held on May 16–18, 2007
Vilnius Lithuania
PROCEEDINGS
Of the 9th International Conference

DISCRETE OPTIMIZATION PROBLEMS OF THE STEEL BAR STRUCTURES

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Received ; accepted

Abstract. In this paper there are considered the optimal design problems of the elastic and elastic-plastic bar structures. These problems are formulated as nonlinear discrete optimization problems. The cross-sections of the bars are designed from steel rolled profiles. The mathematical models of the problems, including the structural requirements of the strength, stiffness and stability, are formulated in the terms of finite element method. The stated nonlinear optimization problems are solved by the iterative method, where each iteration comprises of the selection of the cross-sections of the bars from the assortment and solution of linear problems of the discrete programming. The requirement of discrete cross-sections is ensured by the branch and bound method.

Keywords: elastic and elastic-plastic steel bar structures, discrete optimization, finite element method, mathematical programming

1. Introduction

For the purpose of saving material, the structures are designed by applying the methods of optimization [1-13]. The various algorithms for nonlinear optimization problems of structures are recently created: specific genetic [3-5], discrete optimization [6] and others optimization algorithms [7-11]. The solution algorithms for nonlinear optimization problems are not as universal as the latter for the linear problems. They are mostly dedicated to solve particular type of the problems. Furthermore, the problem of convergence of finding optimal solution occurs frequently, while they are applying. Therefore, nonlinear optimization problems frequently are solved by using the approximation technique when the linear programming problem is solved in each iteration. This method is applied in the paper [12], which is dedicated for the optimization of elastic structures. While designing the structures, an additional economy of the structural material is received for the structures with the plastic deformations in respect to optimal ones with the elastic deformations. However, the optimization problems of elastic-plastic structures [3,8,13] where are evaluated not only the strength, but also stiffness and stability requirements, are complex nonlinear programming problems and realization of them is complicated. In this paper design problems of the elastic and elastic-plastic steel structures are formulated as nonlinear optimization problems. Their mathematical models are created by using finite element method. In these models there are evaluated the conditions of strength, stiffness and stability [14]. The cross-sections are designed from standard steel rolled profiles. The formulated nonlinear optimization problems are solved by the iterative method where each iteration comprises of selection of the cross-sections of the bars from the assortment and solution of linear problems of the discrete programming. The requirement of discrete cross-sections is ensured by the branch and bound method.

2. The volume minimization problem for elastic structures

Mathematical models. There is considered the bar structure loaded by load combinations $v = 1, 2, ..., p$, which bars designed from steel rolled profiles set $\mathbf{\Pi}$. Let the vector $\mathbf{A}_0$ denote the structural bars cross-sectional areas and $\mathbf{F}_v$, $\mathbf{S}_v$, $\mathbf{u}_v$ define the load, internal forces and displacements of v-th load combination, respectively. Then the volume (mass) minimization problem for the elastic structure is expressed by the following mathematical model:

find

$$\min \ f = \mathbf{L}^T \mathbf{A}_0$$

subject to

$$[A]\mathbf{S}_v = \mathbf{F}_v, \quad [\overline{D}]\mathbf{S}_v - [A]^T \mathbf{u}_v = \mathbf{0}, \quad (1)$$
$$[G]\mathbf{A}_0 - [\overline{\Phi}]\mathbf{S}_v \geq \mathbf{0}, \ [E]\mathbf{u}_v \leq \mathbf{u}^+,$$
$$v = 1,2,..., p; \ \mathbf{A}_0 \geq \mathbf{A}_0^-, \ \mathbf{A}_0 \in \mathbf{\Pi}.$$

In this model: equalities – equilibrium and geometrical equations, describing the structural forces and displacements; first inequality – strength and stability conditions; other inequalities – displacements (stiffness) and constructive constraints. $\mathbf{L}$ is the vector of the structural elements lengths. The unknowns of this problem are the vectors $\mathbf{A}_0$, $\mathbf{S}_v$ and $\mathbf{u}_v$. Thus, the objective function of the problem expresses volume and the mass of the structure in the same time. Flexibility matrix $[\overline{D}]$ of the structural elements together with the strength and stability matrix $[\overline{\Phi}]$ depend on unknown $\mathbf{A}_0$. Therefore the model (1) is the nonlinear programming problem: the cross-sections of the structural bars, satisfying the requirements of the minimum volume (mass) of the structure, strength, stiffness and stability, are searching.

By eliminating the internal forces $\mathbf{S}_v = [\overline{D}]^{-1}[A]^T\mathbf{u}_v$ and geometrical equations, this model can be rewrote to the following optimization problem:

find

$$\min \; f = \mathbf{L}^T\mathbf{A}_0$$

subject to

$$[\overline{K}]\mathbf{u}_v = \mathbf{F}_v, \quad [G]\mathbf{A}_0 - [\overline{\Phi}_u]\mathbf{u}_v \geq \mathbf{0}, \qquad (2)$$
$$[E]\mathbf{u}_v \leq \mathbf{u}^+, \quad v = 1,2,...,p;$$
$$\mathbf{A}_0 \geq \mathbf{A}_0^-, \; \mathbf{A}_0 \in \mathbf{\Pi};$$

here $[\overline{\Phi}_u] = [\overline{\Phi}][D]^{-1}[A]^T$; $[\overline{K}] = [A][\overline{D}]^{-1}[A]^T$ is the global stiffness matrix of the structure.

Formulation of the main dependencies. The main dependencies composing the problems (1) and (2) are formulated in the terms of finite element method. For this purpose the structure is divided into the elements (bars) $k = 1,2,...,r$ joined in the nodes. The dependencies of the model (1) can be composed by using the equilibrium finite element method [15], and the model (2) can be created with the help of the equilibrium or geometrically compatible finite element method, because the stiffness matrix $[\overline{K}]$ can be formulated not only of the indicated formula, but also of the stiffness matrices of elements too.

Two equations groups compose *the equilibrium equations* $[A]\mathbf{S}_v = \mathbf{F}_v$:

1) *the equilibrium equations for nodes* describing the relation between the nodal forces of connected into nodes elements and the external forces acting on the nodes;
2) *the equilibrium equations for elements* describing the relation between the nodal forces and acting on the element external load, and are formulated only for elements affected by a distributed load. Expressions of these equations are presented in the papers [12, 15].

The equilibrium equation matrix $[A]$ could be formulated from the coefficients of the equilibrium equations of nodes and elements or from the formula $[A] = [C]^T[\overline{A}]$ [15]; here compatibility matrix $[C]$ describing relation between global displacements of the structural nodes and nodal displacements of elements; $[\overline{A}] = diag[A_k]$ is the quasi-diagonal matrix, which diagonal sub-matrices are composed from the coefficients of the static equations $\mathbf{P}_k = [A_k]\mathbf{S}_k$ of elements.

Flexibility matrix $[\overline{D}] = diag[\overline{D}_k]$ of *geometrical equations* $[\overline{D}]\mathbf{S}_v - [A]^T\mathbf{u}_v = \mathbf{0}$ contains in principal diagonal the flexibility matrices of finite elements $[\overline{D}_k]$. Its coefficients are calculated by formula $d_{ij} = d_k \int_{l_k} H_{ki}(x)H_{kj}(x)dx$, here $H_{ki}(x)$ is the form function of the internal forces; flexibility of the element under tension or compression is $d_k = 1/EA_k$, flexibility of an element under bending is $d_k = 1/EI_k$; E is the elasticity modulus, A_k, I_k are the cross-sectional area and moment of inertia, respectively. Expressions of a matrix $[\overline{D}_k]$ are given in the paper [17].

Strength and stability condition. Strength condition of the element under bending and tension or compression of j-th section is described via inequalities:

$$N_j + c_jM_j - RA_j \leq 0, \; -N_j + c_jM_j - RA_j \leq 0,$$
$$N_j - c_jM_j - RA_j \leq 0, \; -N_j - c_jM_j - RA_j \leq 0. \qquad (3)$$

Here $R = f_{y,d}\gamma_c$; $f_{y,d}$ is the yield strength; γ_c is the partial factor of the exploitation conditions; $c_j = A_j / W_{ej}$; A_j, W_{ej} cross-sectional area and section modulus, respectively.

Furthermore, the bars under compression must satisfy the stability condition

$$-N_j / \varphi_j \leq RA_j \quad \text{or} \quad -N_j / \varphi_j - RA_j \leq 0. \qquad (4)$$

The reduction factor φ for bars under centric or eccentric compression is defined in the national standard of civil engineering [14]. Strength conditions (3) create for all nodes of elements and stability conditions (4) only for the elements under compression. All of them are described via inequality $[G]\mathbf{A}_0 - [\overline{\Phi}]\mathbf{S}_v \geq \mathbf{0}$.

Solution algorithms. The direct solution of the nonlinear discrete programming problems (1) and (2) is fairly complicated. However, their solutions can be found in the iterative process, where in each iteration the cross-sectional profile is selected from the assortment and the linear programming problems solves, which obtain when matrices $[\overline{D}]$, $[\overline{\Phi}]$ and $[\overline{K}]$, $[\overline{\Phi}_u]$ of models (1) and (2) are replaced by matrices $[D]$, $[\Phi]$ and $[K]$, $[\Phi_u]$, which all coefficients are known, because the cross-sections of bars are set. The iterative process is finished, when it is received cross-sectional areas coincide with the previously set ones. For the purpose of minimizing of problem volume it is possible to consider each load case separately and for every one solve such problem:

find

$$\min\ f = \mathbf{L}^T \mathbf{A}_{0v}$$

subject to

$$[A]\mathbf{S}_v = \mathbf{F}_v\,,\ \ [D]\mathbf{S}_v - [A]^T \mathbf{u}_v = \mathbf{0};$$
$$[G]\mathbf{A}_0 - [\Phi]\mathbf{S}_v \ge \mathbf{0}\,,\ \ [E]\mathbf{u}_v \le \mathbf{u}^+;$$
$$\mathbf{A}_{0v} \ge \mathbf{A}_{0,v-1},\quad \mathbf{A}_{0v} \in \mathbf{\Pi} \tag{5}$$

or

find

$$\min\ f = \mathbf{L}^T \mathbf{A}_{0v}$$

subject to

$$[K]\mathbf{u}_v = \mathbf{F}_v;$$
$$[G]\mathbf{A}_0 - [\Phi_u]\mathbf{u}_v \ge \mathbf{0}\,,\ \ [E]\mathbf{u}_v \le \mathbf{u}^+;$$
$$\mathbf{A}_{0v} \ge \mathbf{A}_{0,v-1},\ \ \mathbf{A}_{0v} \in \mathbf{\Pi}. \tag{6}$$

Inequality $\mathbf{A}_{0v} \ge \mathbf{A}_0^-$ for the load cases $v > 1$ is replaced by the condition $\mathbf{A}_{0v} \ge \mathbf{A}_{0,v-1}$. The vector $\mathbf{A}_{0p}$ corresponding to the last load case is the solution of the problems (1) and (2).

Furthermore, the optimization problems (5) and (6) can be solved in two stages:

1) classic problem of structural mechanics is solved i.e. the displacements $\mathbf{u}_v = [K]^{-1}\mathbf{F}_v$ and internal forces $\mathbf{S}_v = [D]^{-1}[A]^T \mathbf{u}_v$ are calculated; for this can be applied the equilibrium or geometrically compatible finite element method and various state-of-the-art computer technology dedicated for this kind of problems;

2) it is determining the vector of strength and stability conditions $\mathbf{S}_{0v} = [\Phi]\mathbf{S}_v$ and solving the minimization problem:

find

$$\min\ f = \mathbf{L}^T \mathbf{A}_0$$

subject to

$$[G]\mathbf{A}_0 \ge \mathbf{S}_{0v},\ \ [G_0]\mathbf{A}_0 \ge [E]u_v,$$
$$\mathbf{A}_0 \ge \mathbf{A}_0^-,\ \ \mathbf{A}_0 \in \mathbf{\Pi}\,,\ \ v = 1, 2, ..., p. \tag{7}$$

Here unknown is the vector $\mathbf{A}_0$, whereas $\mathbf{S}_{0v} = [\Phi]\mathbf{S}_v$. Having software for the internal forces calculations, solution method is easier, because volume of this problem is smaller. It should be noted that it is possible to search for the optimal solution when stability requirements are neglected. But in this case it is necessary to verify if received cross-sections of bars under compression satisfy stability conditions. If they are violated, then cross-sections should be augmented and additional calculation iteration should be performed with including into the mathematical model stability conditions.

Example 1. Let the bar structure, shown in Fig. 1, be loaded by three load cases: I – $p_1 = 16{,}4$ kN/m, $p_2 = 16{,}4$ kN/m; II – $p_1 = 16{,}4$ kN/m, $p_2 = 4$ kN/m; III – $p_1 = 4$ kN/m, $p_2 = 16{,}4$ kN/m. Moreover, the vertical load $F = 27{,}6$ kN and indicated wind load acts in each load case. The optimal cross-sections from steel rolled profiles must be found. Columns and the upper chord are designed from I profiles and others bars from the hollow rectangle tubes. Yield strength $R_y = 275$ MPa, elasticity module $E = 2{,}1 \cdot 10^5$ MPa. Stiffness requirements are described via constraints $u_x \le 5$ cm and $u_y \le 10$ cm, here u_x is the horizontal displacement of top node of the column; u_y is the vertical displacement in the middle of the bottom chord of the truss.

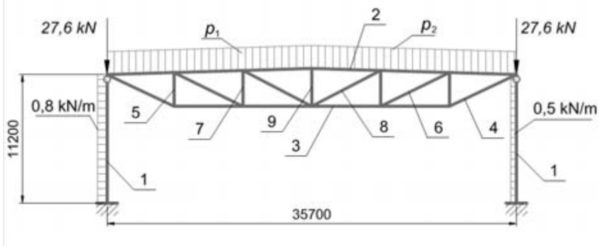

Fig 1. Calculations schema of the framed truss

The columns and the upper chord are calculated as the elements under bending and compression and other ones are calculated as the elements under tension or compression. Cross-sections are selected from the assortment. Initial height of the truss $h = 3{,}3$ m. After optimization it was obtained the following cross-sections: 1 – HEA300; 2 – IPE330; 3 – □ 180x180x6; 4 – □ 150x150x5; 5 – □ 90x90x5; 6 – □ 90x90x4; 7 – □ 70x70x4; 8 - □ 80x80x4; 9 – □ 60x60x5. Total weight of the optimal structure is 5229 kg.

Optimization of the structure is influenced not only by the height of the truss, but also by the web shape and the length of the segments. For this purpose the problems of truss height and web shape were created and considered.

3. Truss height and web shape optimization problems

In this section there are considering and formulating the optimal height and the rational shape of bottom chord of the framed truss, shown in Fig. 1, search problems. Two designed versions are considering: 1) truss with horizontal bottom chord (Fig. 1); 2) truss with parabolic bottom chord (Fig. 2). Height optimization problems of theses trusses are described by such mathematical models:

a) truss with parabolic bottom chord

find

$$\min \ \mathbf{L}^T \mathbf{A}_0$$

subject to

$$[A(\mathbf{l})]\mathbf{S}_v = \mathbf{F}_v, \quad [D(\mathbf{l}, \mathbf{A}_0)]\mathbf{S}_v - [A(\mathbf{l})]^T \mathbf{u}_v = \mathbf{0},$$
$$[G]\mathbf{A}_0 - [\Phi(\mathbf{A}_0)]\mathbf{S}_v \geq \mathbf{0}, \quad [E]\mathbf{u}_v \leq \mathbf{u}^+,$$
$$v = 1,\ 2, ..., p;$$
$$l_j = \left[l_{jx}^2 + \left(y_{j2} - y_{j1}\right)^2\right]^{1/2}, \quad j = 1, 2, ..., s_1;$$
$$l_j = \left[l_{jx}^2 + \left(y_{j2} + y_{0j}\right)^2\right]^{1/2}, \quad j = 1, 2, ..., s_t;$$
$$y_{ji} - a_{ji} f = 0, \ \ \mathbf{A}_0 \geq \mathbf{A}_0^-, \ \ \mathbf{A}_0 \in \mathbf{\Pi};$$

b) truss with horizontal bottom chord

find

$$\min \ \mathbf{L}^T \mathbf{A}_0$$

subject to

$$[A(\mathbf{l})]\mathbf{S}_v = \mathbf{F}_v,$$
$$[D(\mathbf{l}, \mathbf{A}_0)]\mathbf{S}_v - [A(\mathbf{l})]^T \mathbf{u}_v = \mathbf{0},$$
$$[G]\mathbf{A}_0 - [\Phi(\mathbf{A}_0)]\mathbf{S}_v \geq \mathbf{0},$$
$$[E]\mathbf{u}_v \leq \mathbf{u}^+, \quad \mathbf{A}_0 \geq \mathbf{A}_0^-, \quad \mathbf{A}_0 \in \mathbf{\Pi},$$
$$l_j = \left[l_{jx}^2 + \left(f + y_{0j}\right)^2\right]^{1/2},$$
$$j = 1, 2, ..., s_t; \quad v = 1,\ 2, ..., p.$$

Here s_1 is number of bottom chord bars; s_t – number of web bars; f – height of the truss; l_j – length of *j*-th bar, $a_{ji} = 4x_i(l - x_i)/l^2$, l – length of the span. Main unknowns of these problems are cross-sectional areas A_j of bars and height of truss f. There are nonlinear programming problems, which can be solved iteratively.

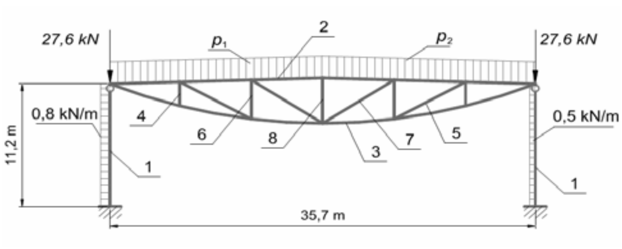

Fig 2. Framed truss with parabolic sketch bottom chord

Example 2. For the analyses of the framed structure in the first example, which is loaded by three prescribed load cases, must be determined: 1) truss rational bottom chord sketch; 2) rational length of the web segment and bars placing; 3) optimal height of the truss. The results of truss investigation are presented in the Fig. 3-5. While performing the analysis of truss bottom chord sketch and web structure it were comparing weight of optimal frame with horizontal and parabolic bottom of truss, while number of truss segments was equal to 6, 8, and 10, and its height $h = 3{,}3 \div 4{,}5$ m. In the Fig. 3 are shown the results of these investigations. The results of calculations showed, that more rational was the truss with horizontal bottom chord.

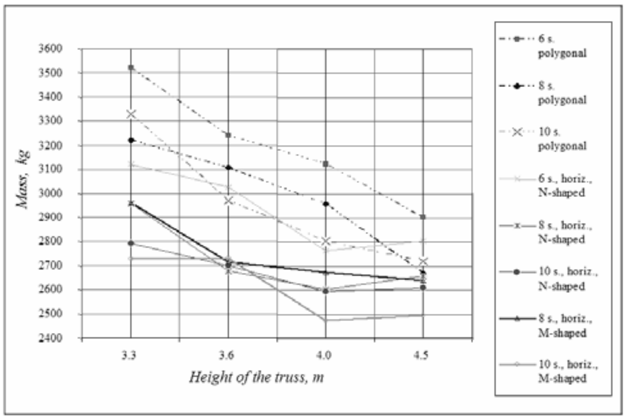

Fig 3. Analysis results of various web and chords shapes

It was investigated N-shaped truss (Fig. 1) and M-shaped truss (Fig. 4). It was determined, that most rational was the structure of the web which was showed in the Fig. 4, and the optimal height of such truss was $h = 4$ m.

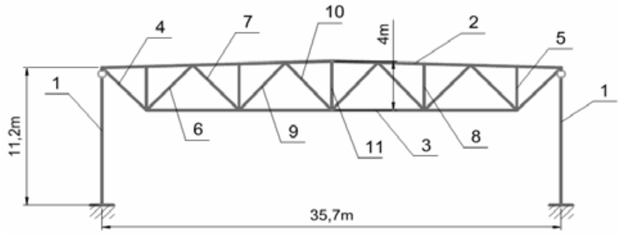

Fig 4. Framed truss with the optimal shape web

In the Fig. 5 are showed chords, web and total mass of truss with optimal shape web dependence on its height. In the Fig. 3 and Fig. 5 are showed only the mass of trusses (mass, equal to 1982 kg, of the columns are not evaluated).

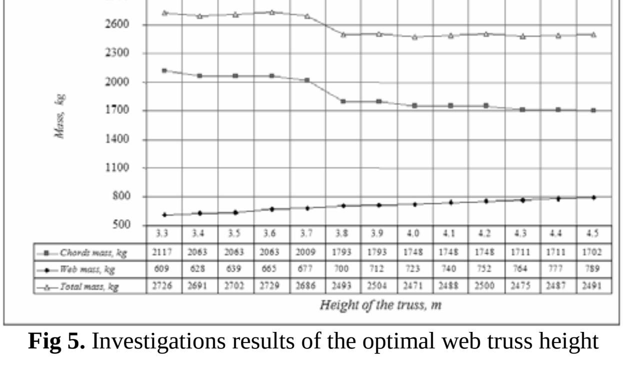

Fig 5. Investigations results of the optimal web truss height

4. The problem of elastic- plastic structure volume optimization

Mathematical models. In the case of the monotonically increasing load the mathematical model of the problem of the minimal volume (mass) elastic-plastic structure can be formulated according to the corresponding optimization model of elastic structure, when the plastic strains $\boldsymbol{\Theta}_p = [\Phi]^T \boldsymbol{\lambda}$ and additional complementary slackness condition are evaluated

$$\boldsymbol{\lambda}^T \left\{ [G]\mathbf{A}_0 - [\overline{\Phi}]^T \mathbf{S} \right\} = 0 \qquad (8)$$

that must correspond plastic multipliers $\boldsymbol{\lambda} \geq \mathbf{0}$. So, referring to the model (1), it is received such, monotonically increasing load acting on elastic-plastic structure, which corresponds to the requirements of the strength, stiffness and stability, mathematical model of optimization problem:

find

$$\min \ \mathbf{L}^T \mathbf{A}_0$$

subject to

$$\begin{gathered} [A]\mathbf{S} = \mathbf{F}, \quad [\overline{D}]\mathbf{S} + [\overline{\Phi}]^T \boldsymbol{\lambda} - [A]^T \mathbf{u} = \mathbf{0}, \\ \boldsymbol{\lambda}^T \left\{ [G]\mathbf{A}_0 - [\overline{\Phi}]\mathbf{S} \right\} = 0, \ \boldsymbol{\lambda} \geq \mathbf{0}, \ [E]\mathbf{u} \leq \mathbf{u}^+, \\ [G]\mathbf{A}_0 - [\overline{\Phi}]\mathbf{S} \geq \mathbf{0}, \quad \mathbf{A}_0 \geq \mathbf{A}_0^-, \quad \mathbf{A}_0 \in \boldsymbol{\Pi}. \end{gathered} \qquad (9)$$

The search of this nonlinear programming problem solution $\mathbf{S}, \mathbf{u}, \boldsymbol{\lambda}, \mathbf{A}_0$ is very difficult. It is especially hardened by the nonlinear conditions (8). That's why it is solved in iteration way, in each iteration selecting cross-sections of bars and solving simpler problem of nonlinear programming, in which only additional complementary slackness conditions are nonlinear. For the purpose of admissible (design) set simplification of the problem and its numerical realization, it is needed to eliminate these conditions from the constraints of the problem. This can be done in two ways - by moving them to the objective function (such possibility is proved in the paper [18] and used in the paper [19]) or eliminating and solving reduced optimization problem. So in each iteration it is possible to solve such problem:

find

$$\min \ f = \mathbf{L}^T \mathbf{A}_0 + \boldsymbol{\lambda}^T \{ [G]\mathbf{A}_0 - [\Phi]\mathbf{S} \}$$

subject to

$$\begin{gathered} [A]\mathbf{S} = \mathbf{F}, \quad [D]\mathbf{S} + [\Phi]^T \boldsymbol{\lambda} - [A]^T \mathbf{u} = \mathbf{0}, \\ [G]\mathbf{A}_0 - [\Phi]\mathbf{S} \geq \mathbf{0}, \quad \boldsymbol{\lambda} \geq \mathbf{0}, \quad [E]\mathbf{u} \leq \mathbf{u}, \\ \mathbf{A}_0 \geq \mathbf{A}_0^-, \quad \mathbf{A}_0 \in \boldsymbol{\Pi} \end{gathered} \qquad (10)$$

or

find

$$\min \ f = \mathbf{L}^T \mathbf{A}_0$$

subject to

$$\begin{gathered} [A]\mathbf{S} = \mathbf{F}, \quad [G]\mathbf{A}_0 - [\Phi]\mathbf{S} \geq \mathbf{0}, \\ [D]\mathbf{S} + [\Phi]^T \boldsymbol{\lambda} - [A]^T \mathbf{u} = \mathbf{0}, \quad \boldsymbol{\lambda} \geq \mathbf{0}, \\ [E]\mathbf{u} \leq \mathbf{u}^+, \quad \mathbf{A}_0 \geq \mathbf{A}_0^-, \quad \mathbf{A}_0 \in \boldsymbol{\Pi}. \end{gathered} \qquad (11)$$

In the first case it is received the problem with nonlinear objective function and liner constrains, and in the second case - the reduced linear programming problem (RLPP). It's understandable, that while solving RLPP, the condition $\lambda_j \left\{ [G_j] A_0 - [\Phi_j] S \right\} = 0$ of some calculated section won't be satisfied. Therefore in this case for defining the optimal solution it is needed to apply the method of branch and bound, setting additional constraints $\lambda_j \leq 0$. for the recent sections.

Example 3. It is needed to set the cross-sections of the bars of the steel rolled profiles of the optimal framed structure, which calculations scheme is exampled in the Fig. 1. The height of the truss is $h = 3{,}3$ m.
The columns and the upper chord of the truss are designed from I profiles, and other bars - from rectangle profile tube. The yield strength of the metal $R_y = 275$ MPa, elasticity module $E = 2{,}1 \cdot 10^5$ MPa. The requirements of the strength is described via constraints $u_x \leq 5$ cm and $u_y \leq 10$ cm; here u_x – horizontal displacement of columns top node, u_y – vertical displacement of truss bottom chord middle node.

Frame bars optimal cross-sections were determined with the help of the branch and bound method by solving reduced nonlinear programming problems. It were received such cross-sections of the bars: 1 – HEA300; 2 – IPE330; 3 – □ 180x180x6; 4 – □ 140x140x5; 5 – □ 90x90x5; 6 – □ 90x90x4; 7 – □ 70x70x4; 8 – □ 80x80x4; 9 – □ 60x60x5. This solution show, that while designing structure, in which it is allowed plastic deformations, it is possible to reduce only tension 4-th bar cross-section. Minimal mass of the optimal elastic-plastic structure $f = 5178$ kg is only 51 kg smaller than the mass of the optimal elastic structure.

5. Conclusions

1. The problems of the steel structures designing are formulated as nonlinear optimization problems. It is demonstrated, that elastic and elastic-plastic structures designing from rolled profiles problems are nonlinear discrete optimization problems, which solutions can be found in the iterative way applying branch and bound method and linear programming.
2. It is proposed three algorithms of optimal bars structures designing, which relations can be formulated applying the methods of equilibrium and geometrically compatible finite elements.

3. It was formulated the problem of truss optimal height determination and, performed calculations it was established, that most rational is 4 m height, i.e. $1/9 \cdot l$ truss (l - length of the span).
4. While performed analysis of the bottom chord sketch, as it were various height of the truss, it was determined that more rational is the truss with parallel bottom chord (Fig. 1), comparing with the truss which bottom chord was form of quadratic parabola (Fig. 2).
5. While fulfilling the analysis of the truss web form and density it was determined, that most rational is the triangle web with vertical bars (Fig. 3), while length of segment is 3,6 m or $1/10 \cdot l$.
6. Elastic-plastic framed structure analysis confirmed statement, that often optimal structure project is determined not by the strength, but stiffness, stability and structural requirements.

References

1. Banichuk, N. V. Introduction to the optimization of the structures. Moscow, 1986. 302 p. (in Russian).
2. Majid, K.I. Optimum design of structures. Moscow, 1979. 237 p. (in Russian).
3. Hayalioglu, M.S. Optimum design of geometrically non-linear elastic-plastic steel frames via genetic algorithm. *Computers & Structures*, Vol 77, 2000, p. 527-538.
4. Hayalioglu, M.S., Degertekin, S.O. Design of non-linear steel frames for stress and displacement constraints with semirigid connections via genetic optimization. *Structural and Multidisciplinary Optimization*, Vol. 27, 2004, p. 259-271.
5. Zheng, Q.Z., Querin, O.M., Barton D. C. Geometry and sizing optimisation of discrete structure using the genetic programming method. *Structural and Multidisciplinary Optimization*, Vol. 31, No 6, 2006, p. 452-461.
6. Gutkowski, W., editor. Discrete Structural optimization. Springer-Verlag, 1997. 250 p.
7. Manickarajah, D., Xie,Y.M., Steven, G.P. Optimum design of frames with multiple constraints using an evolutionary method. *Computers & Structures*, Vol 74, 2000, p. 731-741.
8. Karkauskas, R. Optimization of elastic-plastic geometrically non-linear light-weight structures under stiffness and stability constraints. *Civil Engineering and Management*, Vol. 10, No 2, p. 97-106.
9. Yuge, K., Iwai, N., Kikuchi, N. Optimization of 2D structures subjected to non-linear deformations using the homogenization method. *Structural optimization*, Vol. 17, 1999, p. 286-299.
10. Feng, F.Z., Kim, Y.H., Yang, B.S. Application of hybrid optimization techniques for model updating of rotor shafts. *Structural and Multidisciplinary Optimization*, Vol. 32, No 1, 2006, p. 67-75.
11. Merkevičiūtė, D., Atkočiūnas, J. Optimal shakedown design of metal structures under stiffness and stability constraints. *Journal of Constructional Steel Research*, 62 (2006), p. 1270-1275.
12. Janulevičius, R., Kalanta, S. Optimization of elastic beam structure using linear programming. Material of 8th conference of young Lithuanian scientist "Science - Future of Lithuania", held in Vilnius in March 24-25. Vilnius: Technika, 2005, p. 194-204. (in Lithuanian).
13. Kalanta, S., Grigusevičius, A. Modeling of elastic-plastic beam structures optimization problems by finite element method. The 8th International conference *Modern Building Materials, Structures and Techniques. Selected papers*. Vilnius: Technika, 2004, p. 782-789.
14. Design code STR2.05.08:2005. Design of steel structures. Principal guidelines. Vilnius, 2005. 128 p. (in Lithuanian).
15. Kalanta, S. The equilibrium finite element in computation of elastic structures. *Statyba*, No 1(1). Vilnius: Technika, 1995, p. 25-47. (in Russian).
16. Barauskas, R., Belevičius, R., Kačianauskas, R. Basics of the finite element method. Vilnius: Technika, 2004. 612 p. (in Lithuanian).
17. Kalanta, S. Calculation of the elastic plane bars structures using the finite element method. Questions of theoretical and applied mechanics. *Lithuanian Journal of Computational Mechanics*. No 26, 1983, p. 78-94 (in Russian).
18. Kalanta, S. New formulations of optimization problems of elastoplastic bar structures under displacement constraints. *Mechanika*, No 5(20), 1999, Kaunas: Technologija, p. 9-16 (in Russian).
19. Venskus, A., Atkočiūnas, J. Improved solution algorithm for shakedown optimization problems. Material of 9th conference of young Lithuanian scientist "Science - Future of Lithuania", held in Vilnius in March 29-31. Vilnius: Technika, 2006, p. 265-270 (in Lithuanian).

VILNIAUS GEDIMINO TECHNIKOS UNIVERSITETAS

8-osios Lietuvos jaunųjų mokslininkų konferencijos „LIETUVA BE MOKSLO – LIETUVA BE ATEITIES“, įvykusios Vilniuje 2005 m. kovo 24–25 d., medžiaga

STATYBA

Vilnius 2005

UDK 69 (474.5) (06)
Li-147

8-osios Lietuvos jaunųjų mokslininkų konferencijos „Lietuva be mokslo – Lietuva be ateities“, įvykusios Vilniuje 2005 m. kovo 24-25 d., medžiaga. STATYBA. Vilnius: Technika, 2005. 364 p.

Leidinyje pateikta pranešimų, skaitytų jaunųjų mokslininkų konferencijoje (sekcija "Statyba"), įvykusioje Vilniuje 2005 m. kovo 24-25 d., medžiaga. Pagrindinės konferencijos pranešimų temos – pastatų ir statinių konstrukcijų projektavimas; statybinių konstrukcijų skaičiavimo metodai; statybinės medžiagos ir jų technologija; konstrukcijų optimizavimas ir skaičiavimo metodai; geotechnika; sprendimų priėmimas statyboje; kokybės valdymas; statybos vadyba ir ekonomika; pastatų ūkio valdymas; pastatų ir konstrukcijų gaisrinė sauga; ergonominiai tyrimai ir žmonių sauga.

Ats. redaktorius A. Šneideris

Knygos leidybą rėmė Lietuvos valstybinis mokslo ir studijų fondas

Straipsnių autorių kalba ir stilius netaisyti

VGTU leidyklos „Technika“ 1180 mokslo literatūros knyga

ISBN 9986-05-893-7

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AKMENS VATOS PLOKŠČIŲ GNIUŽDOMOJO ĮTEMPIO NUSTATYMAS PAGAL SKIRTINGUS STANDARTUS

Determination of compression behaviour of mineral wool slabs according to the different standards

Keywords. Mineral wool, Stone wool, Compressive strength, Compressive stress at 10% relative deformation.

Abstract. The present work is devoted to estimate the dependence of the declared value of compressive at 10 % relative deformation of the stone wool slabs on the methodical factors, the different tests of its strength during compression were carried out according to the standards of different countries (EN, DIN, GOST). The influence of the methodical factors on the results of estimation of stone wool slabs' mechanical and deformational characteristics was established by comparing the obtained results.

1. Įvadas

Visi pastatai ir inžinieriniai statiniai turi būti projektuojami ir pastatyti taip, kad atitiktų ne tik saugumo ir funkcionalumo, bet ir optimalios eksploatavimo trukmės, energijos taupymo, aplinkosaugos ir juose

Kaip rodo 3 lentelė didžiausias vienodai pasiskirsčiusios temperatūros pokytis susidaro liepos mėnesį. Todėl kai kurios statiškai neišsprendžiamos konstrukcijos tiltai baigti montuoti vasarą eksploatacijos eigoje gali patirti didžiausius temperatūrinius įtempimus.

3. Išvados

Toliau surašytos išvados suformuluotos Vilniaus miesto metinių oro temperatūrų ekstremumams nuo 1963 iki 2003 metų imtinai.

Metiniai temperatūrų maksimumai, minimumai, maksimalūs metiniai paros oro temperatūros maksimalios amplitudės, metiniai sausio, kovo – gruodžio mėnesių temperatūrų minimumai bei metiniai sausio – gruodžio mėnesių temperatūrų maksimumai gali būti aprašyti normaliuoju skirstiniu. Metinių vasario mėnesio temperatūrų maksimumų normaliuoju skirstiniu aprašyti negalima.

Konstrukcijos sumontuotos liepos mėnesį patiria didžiausias temperatūrines deformacijas.

Straipsnyje pateiktos 50 metų laikotarpio maksimalios oro temperatūros ir RSN 156–94 pateiktos 50 metų maksimalios vidutinės paros oro temperatūros santykinis skirtumas svyruoja nuo 43,8 iki 52,3 %.

Straipsnyje pateiktos 50 metų laikotarpio minimalios oro temperatūros ir RSN 156–94 pateiktos 50 metų minimalios vidutinės paros oro temperatūros santykinis skirtumas svyruoja nuo 25,5 iki 43,2 %.

Straipsnyje pateiktos ir RSN 156–94 pateiktos 50 metų laikotarpio maksimalios paros oro temperatūros amplitudės santykinis skirtumas svyruoja nuo 24,9 iki 35,4 %.

Literatūra

1 ENV 1991-2-5. Eurocode 1 - Basis of design and actions on structures Part 2.5: Thermal actions. 1997. 62 p.
2 RSN 156–94. Building climatology (Statybinė klimatologija). Vilnius, Lietuvos Respublikos statybos ir urbanistikos ministerija, 1995. 136 p.
3 Zavarina M.V. Building climatology (Строительная климатология). Leningrad: Gidrometeoizdat, 1976. 312 p. (in Russian).
4 Montgomery D. C.; Runger G. C. Applied statistics and probability for engineers. New York: John Wiley & Sons, 2003. 709 p.

Padėka. Autoriai dėkoja Lietuvos Hidrometeorologijos centrui už suteiktus oro temperatūrų duomenis.

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PLOKŠTĖS ANALIZĖS NETIESINIO UŽDAVINIO SPRENDIMO PROGRAMA "RŪTA" IR JOS INTEGRACIJOS GALIMYBĖS

Nonlinear analysis problem of plate: computing programme "RUTA" and its integration capabilities

Keywords. Optimal shakedown design, nonlinear programming, circular plates;

Abstract. Universality of the programme "RUTA" (Fortran, Rozen algorithm) of analysis and optimization of shakedown structures is reduced by not fully solved automatic preparation of initial data. That can be solved with the help of programme "SM3". Methodics, introduced by this article, allow integrate programmes "RUTA" and "SM3" using the principles of object oriented programming.

1. Įvadas

VGTU statybinės mechanikos katedroje tampriai plastinių prisitaikančių sistemų optimizavimui ir analizei realizuoti sukurta netiesinių uždavinių matematinių modelių bazė bei jų sprendimui skirtų įvairaus lygio nekomercinių kompiuterinių programų [1, 2]. Viena iš tokių programų yra "RŪTA". Algoritmine kalba Fortran parašytoji programa "RŪTA" skirta prisitaikiusių santvarų, rėmų, lenkiamų plokščių ir lėkštų sferinių kevalų netiesinei analizei ir optimizavimui, kai kintamos kartotinės apkrovos kitimo ribos duotos (atskirais atvejais programa gali realizuoti ir konkrečias žingsnis po žingsnio kintančias apkrovas). Visų programa "RŪTA" sprendžiamų analizės ir optimizacijos uždavinių sprendimo metodas tas pats – Rozeno projektuojamųjų gradientų algoritmas [3]. Programos "RŪTA" (darbe [1] ji įvardijama kaip "SM4") universalumą mažina nepilnai išspręstas pradinių duomenų automatizuotas parengimas: diskretinės konstrukcijos tampraus skaičiavimo įrąžos, pusiausvyros lygčių koeficientų matricos turi būti iš anksto suformuotos, pvz., kitos kompiuterinės programos. Pasitelkus programą "SM3" (aut. prof. St. Kalanta), konstrukcijų diskretizacijai naudojami pusiausviri baigtiniai elementai [1, 4]. Šiame straipsnyje pateikta metodika, leidžianti optimizuoti lenkiamų apvalių plokščių pradinių duomenų formavimą bei jų mainus tarp programų. Tai pasiekiama integruojant programas "RŪTA" ir "SM3" pagal objektinio programavimo principus [5, 6]. Čia integracijos paskirtis buvo tokia: pradinių duomenų kiekio analizės uždaviniui spręsti minimizavimas; automatizuotas duomenų formavimas; automatizuoti duomenų mainai tarp programų, draugiškesnis vartotojo interfeisas. Šiems integracijos tikslams įgyvendinti buvo sukurta nauja programa "INDRĖ-1". Kol kas metodika nerealizuota sudėtingesnių konstrukcijų – kevalų – atvejais, nors principinių ar techninių kliūčių tam nėra. Dar daugiau, ši metodika taip pat gali būti pritaikyta programos "RŪTA" integracijai su kitais, jų tarpe komerciniais, programiniais paketais.

2. Plokštės analizės uždavinio matematinis modelis

Kintama kartotinė apkrova (KKA) – tai sistema jėgų, kurių kiekviena gali nepriklausomai kisti tarp žinomų ribų F_{sup} ir F_{inf}. Jeigu apkrovimo pradžioje dėl plastinio tekėjimo atsiradę liekamieji plokštės momentai M_r kartu su kintamąja momentų dalimi M_e niekur neišeina už leistinųjų ribų, konstrukcija *prisitaiko*. Tai reiškia, jog toliau kintant apkrovai, konstrukcijoje neatsiranda naujų plastinių deformacijų ir ji dirba tartum būtų visiškai tampri. Tokių prisitaikiusių konstrukcijų įtempimų ir deformacijų būvį (analizės uždavinys) nagrinėja prisitaikomumo teorija.

Spendžiant prisitaikiusios konstrukcijos analizės uždavinį yra žinoma konstrukcijos geometrija, jos parėmimo sąlygos, medžiagos fizinis modelis (šiuo atveju - idealiai tampriai plastinis), plokštės skerspjūvio ribinis lenkimo momentas M_0, kintamos kartotinės apkrovos kitimo ribos bei jas

atitinkantys konstrukcijos tampraus skaičiavimo momentai M_e. Ieškomi dydžiai yra liekamosios įrąžos M_r^*. Uždavinio matematinis modelis yra:

rasti $$\min \frac{1}{2} \sum_k M_{rk}^T D_k M_{rk} , \qquad (1)$$

esant sąlygoms

$$AM_r = 0 \qquad (2)$$

$$\varphi_{kl,j} = C_k - \left(M_{ekl,j} + M_{rkl} \right)^T \Phi_{kl} \left(M_{ekl,j} + M_{rkl} \right) \ge 0 ,$$
$$C_k = \left(M_{0k} \right)^2 , \quad k \in K , \quad l \in L , \quad j \in J . \qquad (3)$$

Čia pažymėjimai – pagal darbą [7] . Matematinio programavimo terminais uždavinys (1) - (3) gali būti užrašytas taip:

rasti $$\min \mathcal{F}(x) \qquad (4)$$

esant sąlygoms

$$h_i(x) = 0 , \quad i = 1, 2, \ldots, l . \qquad (5)$$
$$g_i(x) \le 0 , \quad i = 1, 2, \ldots, m , \qquad (6)$$

Čia $\mathcal{F}(x)$ yra uždavinio (1)-(3) tikslo funkcija (1), $h_i(x) = 0$ - pusiausvyros lygtys (2), $g_i(x) \le 0$ - takumo sąlygos (3). Taikant Rozeno algoritmą programa "RŪTA" (o vėliau ir "INDRĖ-1") pateikia uždavinio, dualaus uždaviniui (1)-(3), kintamuosius, t.y plastinius daugiklius λ. Panaudojant λ, skaičiuojamos plastinės deformacijos Θ_p ir liekamieji poslinkiai u_r [2, 7].

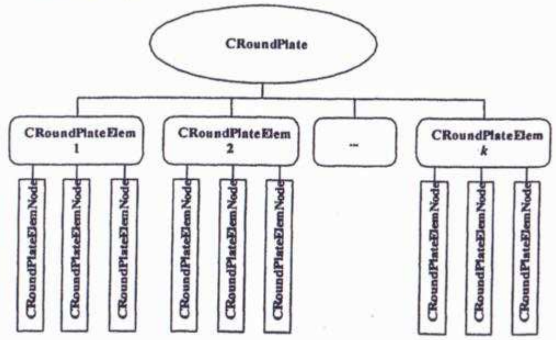
3. Apie programas "RŪTA" ir "SM3"

Programa "RŪTA" skirta spręsti iškilojo netiesinio ir tiesinio matematinio programavimo uždavinius, jų tarpe plokštės analizės uždavinį (1)–(3). Programos "RŪTA" pagrindas - Rozeno algoritmas: tiesiniai arba ištiesinti apribojimai sudaro tiesinę įvairovę, į kurią projektuojama pasirinktoji paieškų kryptis. Kiekviena sprendimo proceso dalis - iteracija susideda iš tokių etapų: leistinojo pradinio vektoriaus (taško) radimas; leistinosios paieškų krypties nustatymas; leistinosios paieškų krypties žingsnio, maksimizuojančio (minimizuojančio) tikslo funkciją, parinkimas. Naujasis vektorius turi išlikti leistinojoje sprendinių kitimo srityje [3].

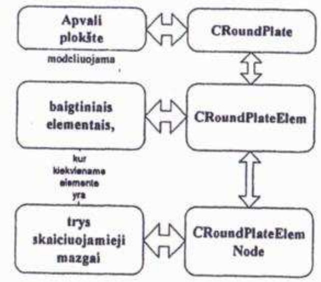
Kaip minėta ankščiau, pradiniams duomenims ruošti yra naudojama programa “SM3”, kurios pagrindą sudaro pusiausvirųjų baigtinių elementų metodas [4]. “SM3” skirta tampriųjų lenkiamų plokščių ir lėkštų sferinių kevalų, veikiamų išorinės apkrovos, įrąžoms ir poslinkiams skaičiuoti bei tampriųjų plastiškųjų lenkiamų plokščių ir lėkštų sferinių kevalų pusiausvyros lygčių koeficientų matricai A formuoti pusiausvirųjų elementų metodu.

4. Apie programų integraciją

Programa “INDRĖ-1” parašyta naudojant C++ ir FORTRAN 90 programavimo kalbas. Programos struktūra gauta paprastesnė ir intuityviai suvokiama, panaudojus objektinio programavimo privalumus [5, 6]. Naudotos Microsoft Visual Studio 6.0 bei Digital Visual Fortran aplinkos. Naudojant programavimo aplinkos komponentus (šiuo atveju MFC žynį), sugeneruotas “INDRĖ-1” grafinis apvalkalas bei jo funkcionalumą realizuojančios klasės. “INDREI-1” pasirinkta SDI (single document interface) architektūra. Objektinio programavimo principams įgyvendinti realūs objektai (apvali plokštė, ją sudarantys pusiausviri baigtiniai elementai) buvo aprašomi atitinkamais duomenų objektais (klasėmis) 1 pav. Atitiktis tarp modeliuojamos konstrukcijos ir duomenų tipų parodyta 2 pav. Pagrindiniai skaičiavimo etapai parodyti 3 pav., o programos “INDRĖ-1” grafinis interfeisas 4 pav.

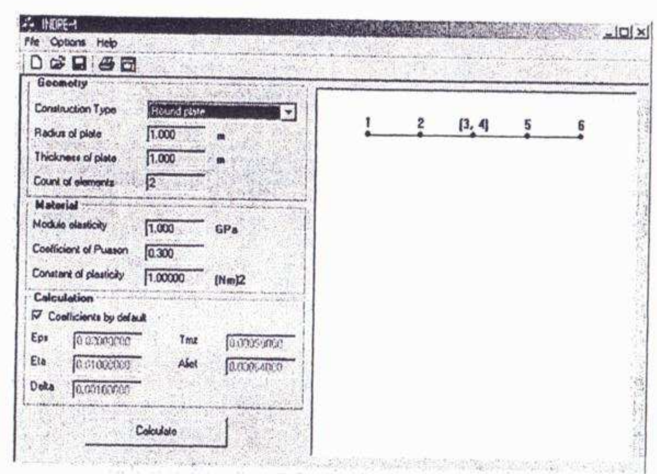


1. pav. Programos “INDRĖ-1” duomenų objektai



2 pav. Atitiktis tarp modeliuojamos konstrukcijos ir duomenų tipų

3 pav. Programos “INDRĖ-1” pagrindiniai skaičiavimo etapai



4 pav. Programos “INDRĖ-1” grafinis vaizdas

5. Išvados

Programų "RŪTA" ir "SM3" integracija atskleidžia galimybę tai išplėsti tarp "RŪTOS" ir komercinių paketų, ypač sprendžiant konstrukcijos parametrų optimizavimo uždavinį. Tuo atveju konstrukcijos pseudotamprų skaičiavimą perkelti paketui, paliekant jam ir gautų optimizavimo rezultatų atvaizdavimą. Tam optimizacijos uždavinių modelis turi būti šiek tiek pertvarkytas.

Literatūra

1. R. Karkauskas, A. Krutinis, J. Atkočiūnas, S. Kalanta, J. Nagevičius. Statybinės mechanikos uždavinių sprendimas kompiuteriais. Vilnius, 1995. 262 p.
2. Ю. Ю. Аткочюнас. Расчет упругопластических систем при повторных нагружениях. Вильнюс, 1994. 144 с.
3. Mokhtar S. Bazaraa, Hanif D. Sherali, C. M. Shetty. Nonlinear Programing Theory and Algorithms. Noida, 2004. 638 p.
4. T. Belytscho, W. K. Liu. Nonlinear Finite Elements for Continua and Structures. New York, 2000. 300 p.
5. Erich Gamma, Richard Helm, Ralph Johnson, John Vlissides. Design Patterns. Boston, 1995. 395 p.
6. B. Meyer. Object oriented software construction. Prentice Hall PTR, 2000. 1296 p.
7. A. Venskus. Lenkiamos plokštės prisitaikymo būvio įražos // 5-osios Lietuvos jaunųjų mokslininkų konferencijos "Lietuva be mokslo- Lietuva be ateities" medžiaga. Vilnius, 2002. P. 298-303.

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ŽUVŲ TAKŲ STATYBOS LIETUVOJE PLĖTROS PERSPEKTYVOS

Development prospects a fish – ladders in Lithuania

Keywords: dam, fish – ladder, fish.

Abstract. Dam construction on the rivers whose are fish migration conventional ways to do a gross harm particularly passing fish, hard shifts fish spawning, their occurrence abatement. There is no question to keep qualitative and quantitative spawning a great many kinds of fish without wild saving. The facts suggest by necessity to seek rationales and ecological safety ways to secure fish migration over hidrotechnical buildings. One of way is fish – loader construction over dam. A fish – loaders design importantly to choose right construction accordant modern requirements.

1. Įvadas

Pastačius užtvankas ant upių, kurios yra tradiciniai žuvų migracijos keliai, daroma didžiulė žala ypač praeivėms ir pusiau praeivėms žuvims, smarkiai pasikeičia žuvų neršto sąlygos, sumažėja jų buvimo sritis. Daugeliui žuvų rūšių neįmanoma išsaugoti kokybinio bei kiekybinio neršto be natūralių neršto sąlygų išsaugojimo. Šie faktai rodo būtinybę ieškoti racionalių ir ekologiškai saugių bei efektyvių būdų žuvų migracijai per hidrotechninius mazgus užtikrinti. Vienas tokių būdų yra žuvitakio įrengimas užtvankoje. Jį projektuojant, svarbu parinkti tinkamą konstrukcija, atitinkančią šiuolaikinius reikalavimus.

VILNIAUS GEDIMINO TECHNIKOS UNIVERSITETAS

9-osios Lietuvos jaunųjų mokslininkų konferencijos „MOKSLAS – LIETUVOS ATEITIS“, įvykusios Vilniuje 2006 m. kovo 29–31 d., pranešimų rinkinys

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VGTU LEIDYKLA TECHNIKA
Vilnius 2006

UDK 69 (474.5) (06)
Mo-62

9-osios Lietuvos jaunųjų mokslininkų konferencijos „Mokslas – Lietuvos ateitis“, įvykusios Vilniuje 2006 m. kovo 29–31 d., pranešimų rinkinys. STATYBA. Vilnius: Technika, 2006. 448 p.

Leidinyje pateikta pranešimų, skaitytų jaunųjų mokslininkų konferencijoje (sekcija „Statyba“), įvykusioje Vilniuje 2006 m. kovo 29–31 d., medžiaga. Pagrindinės konferencijos pranešimų temos – pastatų ir statinių konstrukcijų projektavimas; statybinių konstrukcijų skaičiavimo metodai; statybinės medžiagos ir jų technologija; konstrukcijų optimizavimas ir skaičiavimo metodai; sprendimų priėmimas statyboje; kokybės valdymas; statybos vadyba ir ekonomika; pastatų ūkio valdymas; pastatų ir konstrukcijų gaisrinė sauga; ergonominiai tyrimai ir žmonių sauga.

Ats. redaktorius A. Šneideris

Knygos leidybą rėmė Lietuvos valstybinis mokslo ir studijų fondas

Straipsnių autorių kalba ir stilius netaisyti

VGTU leidyklos „Technika“ 1317 mokslo literatūros knyga

ISBN 9955-28-047-6

TURINYS

SEKCIJA S1
STATYBINĖS MEDŽIAGOS IR DIRBINIAI

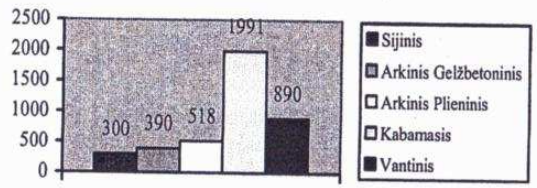
SEKCIJA S2
SAUGOS INŽINERIJA

SEKCIJA S3
STATYBINĖS KONSTRUKCIJOS, KONSTRUKCIJŲ MECHANIKA

Mokslas ir technologijos nestovi vietoje todėl XX amžiaus pabaigoje tarpatramio ilgis pasiekia beveik 2000 metrų. Šiuo metu yra projektuojamas Mesinos tiltas kurio tarpatramio ilgis beveik 4000 metrų.

5. Įvairių konstrukcinių sprendimų tiltų tarpsniai:

Labai didelę reikšmę tilto konstrukcijai turi medžiagų savybės ir jų panaudojimas. Diagramoje (6 pav.) pavaizduota tilto ilgio priklausomybė nuo konstrukcijos tipo. Tuo pačiu atsispindi ir medžiagos įtaka tilto tarpatramiui.



6 pav. skirtingų konstrukcinių sprendinių tiltų ilgiai

6. Išvados

1. Visų konstrukcijų tiltų tarpatramius lėmė naudojamų medžiagų fizinės ir mechaninės savybės.
2. Pastaruoju laikotarpiu pastebimas kabamųjų tiltų platesnis naudojimas, dėl galimybės pilniau išnaudoti pagrindinių juos laikančių elementų savybes.

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PATOBULINTAS PRISITAIKANČIŲ SISTEMŲ OPTIMIZACIJOS UŽDAVINIŲ SPRENDIMO ALGORITMAS

Improved solution algorithm for shakedown optimization problems

Keywords. Optimal shakedown design, Energy principles, Nonlinear programming

Abstract. In this paper authors improve their previously created algorithm for solution of shakedown structures. Initially optimization problem is solved with “soft” rigidity conditions and only after receive of optimal solution, independent from loading history “hard” rigidity conditions are verified. Examples of optimization problems solution of frame with small displacements are presented.

1. Įvadas

Siekiant konstrukcijų projektavimą priartinti prie realių darbo sąlygų įvertinimo, naudinga atsižvelgti ir į galimą elementų plastinį darbą, į apkrovų kintamą kartotinį pobūdį. Tai skatina kurti naujus prisitaikančių sistemų optimizacijos uždavinių matematinius modelius ir jų sprendimo algoritmus. Optimalumo kriterijais čia gali būti maksimali apkrova arba toks ribinių įrąžų pasiskirstymas, kuriam esant yra minimalus konstrukcijos svoris arba kaina. Darbe [1] pateikti bendrieji prisitaikančių sistemų optimizacijos uždavinių matematiniai modeliai. Čia, optimizuojant apkrovos arba ribinių įrąžų pasiskirstymus, reikšmingi tampa konstrukcijos standumo, dažniausiai siejamo su įlinkiais, reikalavimų tenkinimas. Straipsnyje siūlomas patobulintas prisitaikančių sistemų optimizacijos uždavinių sprendimo algoritmas. Iš pradžių sprendžiamas optimizacijos uždavinys su "švelnesnėmis" standumo sąlygomis ir tik po to, gavus optimalų sprendinį, tikrinamos nuo apkrovimo istorijos nepriklausančios "griežtesnės" standumo sąlygos. Matematinio griežtumo sąlygos, sekant darbą [2], įtrauktos į optimizacijos uždavinio tikslo funkciją.

2. Optimizacijos uždavinių bendros sąvokos

Nagrinėjamas idealiai tamprus plastinis rėmas, kurio geometrija yra žinoma. Diskretizacijai naudojami pusiausvyri baigtiniai elementai. Aproksimuojamos tik įrąžos – lenkimo momentai M ir ašinės jėgos N. Kai rėmo elementams naudojami I ar H tipo skrerspjūviai, galima naudoti tiesines takumo sąlygas:

$$|M| + c|N| \le M_0, \; c = \frac{M_0}{N_0} \tag{1}$$

Apkrova kintama kartotinė ir yra charakterizuojama nuo laiko t nepriklausančiomis viršutinėmis ir apatinėmis jėgų kitimo ribomis $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$. Sistema prisitaiko prie kintamos kartotinės apkrovos, jei bet kuriai apkrovimo istorijai egzistuoja statiškai galimos, nepriklausančios nuo laiko t, liekamosios įrąžos $\boldsymbol{S}_r = (\boldsymbol{M}_r, \boldsymbol{N}_r)$.

3. Apkrovos optimizacijos uždavinio matematinis modelis

Apkrovos optimizacijos uždavinys formuluojamas taip: ieškomos apkrovos kitimo ribos $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$, tenkinančios užduotą optimalumo kriterijų $\max \{\boldsymbol{T}_{sup}^T \boldsymbol{F}_{sup} - \boldsymbol{T}_{inf}^T \boldsymbol{F}_{inf}\}$ bei rėmo stiprumo ir standumo sąlygas, kai yra žinomi elementų skerspjūviai bei jų stiprumінės charakteristikos M_0, N_0:

rasti $\max \mathcal{F}(\boldsymbol{F}_{sup}, \boldsymbol{F}_{inf}, \lambda_j) =$

$$\left\{ \boldsymbol{T}_{sup}^T \boldsymbol{F}_{sup} - \boldsymbol{T}_{inf}^T \boldsymbol{F}_{inf} - \sum_{j=1}^{p} \lambda_j^T \left[\boldsymbol{M}_0 - [\Phi] \left([G][\Phi]^T \sum_{j=1}^{p} \lambda_j^T + [\alpha] \boldsymbol{F}_j \right) \right] \right\}, \tag{2}$$

kai
$$\varphi_j = \boldsymbol{M}_0 - [\Phi] \left([G][\Phi]^T \sum_{j=1}^{p} \lambda_j^T + [\alpha] \boldsymbol{F}_j \right), \tag{3}$$

$$\lambda_j \ge 0, \tag{4}$$

$$\boldsymbol{F}_{sup} \ge 0, \; -\boldsymbol{F}_{inf} \ge 0, \tag{5}$$

$$\boldsymbol{u}_{r,\min} \le_{\min}^{\max} [H][\Phi]^T \sum_{j=1}^{p} \lambda_j^T \le \boldsymbol{u}_{r,\max}. \tag{6}$$

Uždavinio (2)-(6) nežinomieji yra apkrovų ribų vektoriai $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$, bei plastiniai daugikliai λ_i. Apkrovos svarbą įvertina optimalumo kriterijaus svorio koeficientų vektoriai $\boldsymbol{T}_{sup}$ ir $\boldsymbol{T}_{inf}$. Sąlygos (6) yra standumo apribojimai, kai ribojamos tik liekamųjų poslinkių reikšmės (čia $\boldsymbol{u}_{r,\min}$, $\boldsymbol{u}_{r,\max}$ - normomis leistinos liekamųjų poslinkių kitimo ribos). Tiesinės takumo sąlygos užrašomos pasitelkus matricą $[\Phi]$, liekamosioms ir tampriosioms įrąžoms skaičiuoti naudojamos influentinės matricos $[G]$ ir $[\alpha]$, o liekamiesiems poslinkiams - $[H]$.

4. Ribinių įrąžų optimizacijos uždavinio matematinis modelis

Ribinių įrąžų optimizacijos uždavinys formuluojamas taip: ieškomos ribinės įrąžos $\boldsymbol{M}_0$, tenkinančios užduotą optimalumo kriterijų $\min \boldsymbol{L}^T \boldsymbol{M}_0$ bei rėmo stiprumo ir standumo sąlygas, kai yra žinomos apkrovos kitimo ribos $\boldsymbol{F}_{sup}$, $\boldsymbol{F}_{inf}$:

rasti $\min \mathcal{F}(\boldsymbol{M}_0, \lambda_j) =$

$$\left\{ L^T M_0 - \sum_{j=1}^{p} \lambda_j^T \left[M_0 - [\Phi]\left([G][\Phi]^T \sum_{j=1}^{p} \lambda_j^T + [\alpha] F_j \right) \right] \right\}, \tag{7}$$

kai

$$\varphi_j = M_0 - [\Phi]\left([G][\Phi]^T \sum_{j=1}^{p} \lambda_j^T + [\alpha] F_j \right), \quad 8)$$

$$\lambda_j \geq 0, \tag{9}$$

$$M_{0\max} - M_0 \geq 0, \; M_0 - M_{0\min} \geq 0, \tag{10}$$

$$u_{r,\min} \leq_{\min}^{\max} [H][\Phi]^T \sum_{j=1}^{p} \lambda_j^T \leq u_{r,\max}. \tag{11}$$

Uždavinio (7)-(11) nežinomieji yra ribinės įrąžos M_0, bei plastiniai daugikliai λ_i. Elementų ribinių įrąžų svarba tikslo funkcijoje įvertinama vektoriaus L pagalba, o maksimalios ir minimalios ribinių įrąžų reikšmės užrašomos vektoriais $M_{0\max}$ ir $M_{0\min}$. Matricų $[\Phi]$, $[G]$, $[\alpha]$, $[H]$ prasmė tokia pat kaip uždavinyje (2) – (6).

5. Optimizacijos uždavinių sprendimo ypatumai

Kaip matome iš uždavinių (2) – (6) ir (7) – (11) matematinių modelių, standumo sąlygos (6) ir (11) yra analogiškos. Jos optimizacijos uždavinius skatina spręsti etapais, nes prisitaikiusios konstrukcijos deformuotas būvis priklauso nuo apkrovimo istorijos. Uždavinių sprendimo palengvinimui sąlygos (6) ir (11) formuluojamos taip:

$$u_{r,\min} \leq [H][\Phi]^T \sum_{j=1}^{p} \lambda_j^T \leq u_{r,\max}. \tag{12}$$

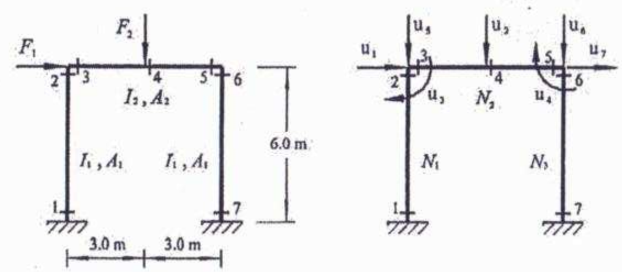
Gavus optimalų sprendinį grįžtama prie sąlygų (6) ar (11) tikrinimo. Jei jos pažeistos, tenka skaičiavimą pakartoti. Tokiu atveju sugriežtinamos $u_{r,\min}$, $u_{r,\max}$ reikšmės.

Optimizacijos uždaviniai (2) – (6) ir (7) – (11) yra netiesinio matematinio programavimo uždaviniai. Sprendimui naudojamas Rozeno

projektuojamųjų gradientų metodas [3]-[5]. Algoritmas patobulintas: matematinio programavimo griežtumo sąlygas įtraukus į tikslo funkciją reikšmingai supaprastėja optimizacijos uždavinių sprendimas [2].

6. Skaitiniai pavyzdžiai

Nagrinėjamas rėmas, kurio kolonoms naudojami HE ir rygeliui IPE tipo profiliai, pagaminti iš plieno (takumo riba σ_y =240 MPa, tamprumo modulis $E=2\cdot10^5$ MPa, 1 pav.) ir yra veikiamas kartotinės apkrovos $0 \leq F_1 \leq F_{1,sup}$, $0 \leq F_2 \leq F_{2,sup}$.



1. pav. Rėmo skaičiuojamoji schema ir jo diskretinis modelis

Apkrovos optimizacijos uždavinys $\max(F_{1,sup} + F_{2,sup})$ sprendžiamas pagal matematinį modelį (2)-(6). Šiuo atveju rėmo elementų skerspjūviai yra žinomi ir kolonoms naudojami HE240A, o rygeliui – IPE400.

Neįvertinus standumo reikalavimų (6) gauta $F_{1,sup}^{*}$=65.17 kN ir $F_{2,sup}^{*}$=281.68 kN t.y. $\max(F_{1,sup} + F_{2,sup})$=346.85 kN.

Įvertinus standumo reikalavimus $0 \leq u_{r,1} \leq 20$ mm, $0 \leq u_{r,2} \leq 25$ mm, gauta $F_{1,sup}^{*}$=67.81 kN ir $F_{2,sup}^{*}$=259.21 kN ($\max(F_{1,sup} + F_{2,sup})$=327.02 kN).

Ribinių įrąžų optimizacijos uždavinys min $L^T M_0$ sprendžiamas pagal matematinį modelį (7)-(11). Apkrovos kitimo ribos yra žinomos: $F_{1,sup}$=67.81 kN ir $F_{2,sup}$=259.21 kN. Ribinių įrąžų santykis $c = \frac{M_0}{N_0}$ kolonoms imamas 0.0969 m, o rygeliui 0.15476 m.

Įvertinus standumo reikalavimus $0 \le u_{r,1} \le 10$ mm, $0 \le u_{r,2} \le 15$ mm, gauta: kolonoms M_0^*=159.77 kNm ir rygeliui M_0^*= 339.47 kNm ($\min L^T M_0 = 3954.017$ kNm^2).

7. Išvados

Matematinio programavimo griežtumo sąlygos, įeinančios į optimizacijos uždavinių matematinius modelius, neleidžia įvertinti konstrukcijos pjūvių nusikrovimo reiškinio. Todėl pagrindinis optimizacijos uždavinys turi būti sprendžiamas etapais. Pirmame etape ignoruojamas nusikrovimo reiškinys, antrame etape, taikant tiesinį matematinį programavimą, tikrinamos ekstreminės liekamųjų poslinkių reikšmės. Tiesinio programavimo uždavinys sudarytas, pasitelkus liekamųjų deformacijų darnos lygtis.

Literatūra

1. Atkočiūnas J. Mathematical models of optimization problems at shakedown. Mechanics Research Communications, 26, No 3, 1999, p. 319-326.
2. S. Kalanta. Naujos tampriųjų-plastinių strypinių konstrukcijų formuluotės esant poslinkių apribojimams. Mechanika, 1999, Nr. 5(20), Kaunas, p. 9-16. (rusų k.).
3. A. Venskus. Plokštės analizės netiesinio uždavinio sprendimo programa "Rūta" ir jos integracijos galimybės // 8-osios Lietuvos jaunųjų mokslininkų konferencijos "Lietuva be mokslo- Lietuva be ateities" medžiaga. Vilnius, 2005, p. 277-282.
4. A. Venskus. Lenkiamos plokštės prisitaikymo būvio įražos // 5-osios Lietuvos jaunųjų mokslininkų konferencijos "Lietuva be mokslo- Lietuva be ateities" medžiaga. Vilnius, 2002, p. 298-303.
5. Mokhtar S. Bazaraa, Hanif D. Sherali, C. M. Shetty. Nonlinear Programing Theory and Algorithms. Noida, 2004, 638 p.

9-oji Lietuvos jaunųjų mokslininkų konferencija
„Mokslas – Lietuvos ateitis"
Sekcija **STATYBA**, 2006 m. kovo 29-31 d., Vilnius

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INVESTIGATIONS ON CONTINUOUS COMPOSITE BEAMS WITH LARGE WEB OPENINGS

1. Introduction

Continuous composite beams can be designed according to the yield hinge method, which is explained by Bode [1] in detail. With this method the plastic reserves of cross section and system can be used to full capacity. This design method generates many questions if beams with large web openings are regarded. These questions are not clarified so far. At the Institute for Concrete Structures and Structural Design of the Technical University of Kaiserslautern a research project has started to solve these problems. Within this project two large-scale tests were arranged until now. Furthermore a comprehensive parameter study was accomplished to solve the unexplained questions.

In this paper problems to the mentioned topic are explained and the experimental and computed investigations, which were arranged, are presented.